

2. Prove the following identities, in which m denotes an arbitrary positive integer:

$$\cos mz = \cos^m z - \binom{m}{2} \cos^{m-2} z \sin^2 z + \binom{m}{4} \cos^{m-4} z \sin^4 z - \dots$$

(the last term of the right side is $(-1)^{m/2} \sin^m z$ or $(-1)^{(m-1)/2} m \cos z \sin^{m-1} z$, depending on whether m is an even or odd number);

$$\sin mz = m \cos^{m-1} z \sin z - \binom{m}{3} \cos^{m-3} z \sin^3 z + \dots$$

(the last term of the right side is $(-1)^{(m+2)/2} m \cos z \sin^{m-1} z$ or $(-1)^{(m-1)/2} \sin^m z$, depending on whether m is an even or odd number);

$$2^{m-1} \cos^m z = \cos mz + \binom{m}{1} \cos(m-2)z + \binom{m}{2} \cos(m-4)z + \dots$$

(the last term of the right side is $\frac{1}{2} \binom{m}{m/2}$ or $\binom{m}{(m-1)/2} \cos z$, depending on whether m is an even or odd number);

the analogous formula for $\sin^m z$: for even m ,

$$(2i)^m \sin^m z = 2 \cos mz - 2 \binom{m}{1} \cos(m-2)z + 2 \binom{m}{2} \cos(m-4)z - \dots + (-1)^{m/2} \binom{m}{m/2},$$

and for odd m ,

$$(2i)^{m-1} \sin^m z = \sin mz - \binom{m}{1} \sin(m-2)z + \binom{m}{2} \sin(m-4)z + \dots + (-1)^{(m-1)/2} \binom{m}{(m-1)/2} \sin z.$$

[Hint. Take, first, z real in the identities

$$\cos mz + i \sin mz = (\cos z + i \sin z)^m, \quad 2^m \cos^m z = (e^{iz} + e^{-iz})^m,$$

which follow directly from formulae (8.2) and (8.3), expand the right sides by Newton's binomial theorem and equate the real and imaginary parts of both sides. The generalization to complex values of z follows from the theorem stating that a power series which vanishes for real values of the variable, vanishes identically.]

3. Show that the trigonometric expression

$$2^n \sin^n \theta + \binom{n}{1} 2^{n-1} \sin^{n-1} \theta \cos \left(\theta + \frac{\pi}{2} \right) + \\ + \binom{n}{2} 2^{n-2} \sin^{n-2} \theta \cos 2 \left(\theta + \frac{\pi}{2} \right) + \dots + \cos n \left(\theta + \frac{\pi}{2} \right)$$

is equal to $(-1)^{n/2} \cos n\theta$ or $(-1)^{(n-1)/2} \sin n\theta$, depending on whether n is an even or odd number.

[Hint. Consider the expression $\{2 \sin \theta + \exp i(\theta + \pi/2)\}^n$.]

4. Substituting $w = \sin z$ show that:

(a) $\cos mz = F_1(w)$, $\sin mz / \cos z = G_1(w)$, for every even positive integral value of m ,

(b) $\cos mz / \cos z = F_2(w)$, $\sin mz = G_2(w)$, for every odd positive integral value of m ,

where $F_1(w)$ and $G_2(w)$ are polynomials in w of degree m , and $G_1(w)$ and $F_2(w)$ are of degree $m-1$.

[Hint. Cf. exercise 2.]

5. Show that the roots of the polynomials $F_1(w)$, $G_1(w)$, $F_2(w)$, and $G_2(w)$, exercise 4, are respectively:

$$\begin{aligned} & \pm \sin \frac{\pi}{2m}, \quad \pm \sin \frac{3\pi}{2m}, \quad \dots, \quad \pm \sin \frac{(m-1)\pi}{2m}; \\ 0, \quad & \pm \sin \frac{\pi}{m}, \quad \pm \sin \frac{2\pi}{m}, \quad \dots, \quad \pm \sin \frac{m-2}{2} \frac{\pi}{m}; \\ & \pm \sin \frac{\pi}{2m}, \quad \pm \sin \frac{3\pi}{2m}, \quad \dots, \quad \pm \sin \frac{(m-2)\pi}{2m}; \\ 0, \quad & \pm \sin \frac{\pi}{m}, \quad \pm \sin \frac{2\pi}{m}, \quad \dots, \quad \pm \sin \frac{m-1}{2} \frac{\pi}{m}. \end{aligned}$$

Derive from this the following formulae:

(a) for even m :

$$\begin{aligned} \cos mz &= \left(1 - \frac{\sin^2 z}{\sin^2 \frac{\pi}{2m}} \right) \left(1 - \frac{\sin^2 z}{\sin^2 \frac{3\pi}{2m}} \right) \dots \left(1 - \frac{\sin^2 z}{\sin^2 \frac{(m-1)\pi}{2m}} \right), \\ \frac{\sin mz}{\cos z} &= m \sin z \cdot \left(1 - \frac{\sin^2 z}{\sin^2 \frac{\pi}{m}} \right) \left(1 - \frac{\sin^2 z}{\sin^2 \frac{2\pi}{m}} \right) \dots \left(1 - \frac{\sin^2 z}{\sin^2 \frac{m-2}{2} \frac{\pi}{m}} \right); \end{aligned}$$

(β) for odd m :

$$\frac{\cos mz}{\cos z} = \left(1 - \frac{\sin^2 z}{\sin^2 \frac{\pi}{2m}} \right) \left(1 - \frac{\sin^2 z}{\sin^2 \frac{3\pi}{2m}} \right) \dots \left(1 - \frac{\sin^2 z}{\sin^2 \frac{(m-2)\pi}{2m}} \right),$$

$$\sin mz = m \sin z \left(1 - \frac{\sin^2 z}{\sin^2 \frac{\pi}{m}}\right) \left(1 - \frac{\sin^2 z}{\sin^2 \frac{2\pi}{m}}\right) \cdots \left(1 - \frac{\sin^2 z}{\sin^2 \frac{m-1}{2} \frac{\pi}{m}}\right).$$

6. Substituting z/m for z in formulae (α) of exercise 5 and passing to the limit as $m \rightarrow +\infty$, derive the following expansions of the cosine and the sine in infinite products:

$$\begin{aligned} \cos z &= \left(1 - \frac{z^2}{(\pi/2)^2}\right) \left(1 - \frac{z^2}{(3\pi/2)^2}\right) \left(1 - \frac{z^2}{(5\pi/2)^2}\right) \cdots, \\ \sin z &= z \left(1 - \frac{z^2}{\pi^2}\right) \left(1 - \frac{z^2}{4\pi^2}\right) \left(1 - \frac{z^2}{9\pi^2}\right) \cdots \end{aligned}$$

The preceding products are uniformly convergent in every circle of finite radius, *i. e.* almost uniformly in the entire plane (by the convergence of the infinite product $A_1 \cdot A_2 \cdots A_n \cdots$ we here mean the convergence of the sequence of partial products $P_1 = A_1$, $P_2 = A_1 \cdot A_2$, ..., $P_n = A_1 \cdot A_2 \cdots A_n$, ...; a more detailed discussion of the notion of limit of an infinite product will be given in Chapter VII).