1. DISTRIBUTIONS

A distribution is a linear functional

$$L : \mathcal{D} \to \mathbb{R} \quad (\text{or } \mathbb{C}) \qquad \text{where} \qquad \mathcal{D} = C_c^\infty(\mathbb{R}^n)$$

which is linear and continuous.

1.1. Topology on $\mathcal{D}$.

Let $K \subset \mathbb{R}^n$ be a compact set. Define
$\mathcal{D}_K := \{f \in \mathcal{D} : f \equiv 0 \text{ outside } K\}$

$$\|f\|_{K,m} = \sup_{x \in K, |s| \le m} |D^s f(x)|.$$

Then $f_n \to f$ means
(1) $supp f_n \subseteq K$ same K for all n
(2) for any D^s, $D^s f_n \to D^s f$ uniformly.

T is continuous means for every K there is C_K and $m > 0$ so that

$$|T(\psi)| \le C\|\psi\|_{K,m} \quad \forall\ \psi \in \mathcal{D}_K.$$

1.2. The distribution x_+^λ.

$$T_\lambda(x) := \int_0^\infty x^\lambda \varphi(x)dx$$

is well defined for $\lambda \in \mathbb{C}$ such that $Re\lambda > -1$.

$$T_\lambda(x) = \int_0^\infty x^\lambda \varphi(x)dx = \frac{x^{\lambda+1}}{\lambda+1}\varphi(x)\Big|_0^\infty - \frac{1}{\lambda+1}\int_0^\infty x^{\lambda+1}\varphi'(x)dx$$

$$du = x^\lambda dx \quad v = \varphi(x)$$

$$u = \frac{x^{\lambda+1}}{\lambda+1} \quad dv = \varphi'(x)dx$$

So

$$T_\lambda(x) = \frac{1}{\lambda+1}T'_{\lambda+1}$$

The right hand side makes sense for $Re(\lambda+1) > -1$ or $Re\lambda > -2$.

In this way we can extend T_λ to $Re\lambda > -2$, but there is a "pole" at $\lambda = -1$.

$$\lim_{\lambda \to -1}(\lambda+1)T_\lambda = T'_0 = -\int_0^\infty (\varphi'(x))dx = -\varphi(x)\Big|_0^\infty = \varphi(0)$$

So δ_0 is the residue of T_λ at $\lambda = -1$.

1.3. **Distributions with support at** 0.

Definition. *We say a distribution* T *vanishes in an open set* $\mathcal{U}$, *if* $T(\varphi) = 0$ *for any function* $\varphi \in C_c^\infty(\mathbb{R})$, *with support in* $\mathcal{U}$.

The support of a distribution is the smallest closed set F *such that* T *vanishes in the open set* $\mathcal{U} = \mathbb{R}\backslash F$.

Theorem. *A distribution* T *such that supp*$T = \{0\}$ *is a derivative of* δ_0.

Proof. Let ψ be a function in $\mathcal{D}$ which is $\equiv 1$ on $(-1/2, 1/2)$, and vanishes outside $(-1, 1)$. Then write $\psi_\varepsilon(x) = \psi(x/\varepsilon)$. $\psi_\varepsilon(x)$ is identically 1 in $(-\varepsilon, \varepsilon)$. Let $f_\varepsilon = f \cdot \psi_\varepsilon$. Because of the support property,

$$T(f) = T(f_\varepsilon)$$

Indeed this is because $f - f_\varepsilon \equiv 0$ near 0, so $T(f - f_\varepsilon) = 0$. Because of continuity, for any compact K,

$$|T(f)| \leq C \sup_{|j|<k, x\in K} |D^j f|.$$

We will apply this to the functions f_ε, with $\varepsilon \to 0$, so we can take $K = [-1, 1]$. By direct calculation, we see that $D^j f_\varepsilon$ is of the form $|\varepsilon|^{-j} D^j\psi \cdot D^i f$, $|i| + |j| \leq k$.

Assume that $D^j f(0) = 0$ for all $|j| \leq k$. By Taylor's formula,

$$|D^s f_\varepsilon| \leq C\varepsilon^{k+1-|s|}$$

So $T(f) = T(f_\varepsilon)$, and $|T(f_\varepsilon)| \leq C \cdot \varepsilon \to 0$.

Now suppose f is arbitrary.

$$f_k = \Big(\sum_{j\leq k} \frac{f^j(0)}{j!} x^j\Big)\psi \in C_c^\infty$$

and has the same first k derivatives as f. So by the previous argument, $T(f - f_k) = 0$. But then

$$T(f) = T(f_k) = \sum f^{(j)}(0) T\Big(\frac{x^j}{j!}\psi\Big),$$

and the $T\Big(\frac{x^j}{j!}\psi\Big)$ are fixed constants independent of f. Thus we have proved that

$$T(f) = \sum_{j\leq k} c_j D^j(f)(0). \tag{1.3.1}$$

□

2. Fourier Inversion

2.1. Recall that the Fourier transform

$$\mathcal{F}_x(f) := \int e^{i\xi x} f(x)\, dx \tag{2.1.1}$$

takes the Schwartz space $\mathcal{S}$ to itself. We would like to compute its inverse. The main theorem is the following.

Theorem. *The operator*

$$\mathcal{F}_\xi(\phi) := \frac{1}{2\pi}\int e^{-i\xi x}\phi(\xi)\, d\xi$$

satisfies

$$\mathcal{F}_\xi \circ \mathcal{F}_x(f) = f, \qquad \mathcal{F}_x \circ \mathcal{F}_\xi(\phi) = \phi.$$

We will treat one of the equations, the other one is the mirror image. The natural thing to try is to write out the composition, and change the order of integration:
(2.1.2)

$$\mathcal{F}_\xi \circ \mathcal{F}_x(f)(y) = \frac{1}{2\pi}\int e^{-i\xi y}\int e^{i\xi x} f(x)\, dx\, d\xi = \int f(x)\int \frac{e^{i\xi(x-y)}}{2\pi}\, d\xi\, dx.$$

But the inner integral does not make sense. Formally,

$$\int e^{i\xi(x-y)}\, d\xi \tag{2.1.3}$$

is the Fourier transform of the function $\phi(\xi) = \frac{1}{2\pi}$, at $x - y$:

$$\widehat{\phi}(x-y) = \int e^{i(x-y)\xi}\frac{1}{2\pi}\, d\xi. \tag{2.1.4}$$

If we write $\psi_x(\xi) := \frac{1}{2\pi}e^{ix\xi}$, then (2.1.3) is also $\widehat{\psi}(-y)$.

To deal with the inability to just change the order of integration, and evaluate the inner integral, we resort to distributions.

Definition. *Let T be a tempered distribution. We define the Fourier transform $\widehat{T}$ to be*

$$\widehat{T}(f) := T(\widehat{f}).$$

This is well defined, because $\widehat{f} \in \mathcal{S}$. When $T = L_\phi$, this coincides with the usual Fourier transform.
(2.1.5)

$$\begin{aligned}\widehat{L_\phi}(f) &= \int \phi(\xi)\int e^{i\xi x} f(x)\, dx\, d\xi = \int f(x)\int e^{i\xi x}\phi(x)\, dx\, d\xi = \\ &= \int f(x)\widehat{\phi(x)}\, dx.\end{aligned}$$

The change of order of integration is justified because both f and ϕ are in $\mathcal{S}$.

2.2. From this point of view, Fourier inversion comes down to asking

What distribution satisfies the property $\delta_0 = \widehat{f}$?

The equation is the same as $f(0) = T(\widehat{f})$. Given such a T, we can implement Fourier inversion as follows. Given x, define the *translation by x operator* $\mathcal{L}_x$ as

$$\mathcal{L}_x(f)(y) := f(x+y). \tag{2.2.1}$$

Then

$$f(x) = (\mathcal{L}_x(f))(0) = T(\widehat{\mathcal{L}_x(f)}). \tag{2.2.2}$$

On the other hand,

$$\begin{aligned}\widehat{\mathcal{L}_x(f)}(\xi) = \int e^{i\xi y}\mathcal{L}_x(f)(y)\ dy = \int e^{i\xi y} f(x+y)\ dy = \\ \int e^{i\xi(u-x)} f(u)\ du = e^{-i\xi x}\widehat{f}(\xi).\end{aligned} \tag{2.2.3}$$

So equation (2.2.2) becomes
(2.2.4)

$$f(x) = T(-e^{i\xi x}\widehat{f}(\xi)). \qquad (\ T \text{ acts on the RHS as a function of } \xi).$$

2.3. Let $h_t(x) := e^{-\frac{x^2 t}{2}}$. This function is in $\mathcal{S}$. Then,

$$\lim_{t\to 0} e^{-\frac{x^2 t}{2}} f(x) = f(x), \qquad |e^{-\frac{x^2 t}{2}} f(x)| \le |f(x)|, \tag{2.3.1}$$

so by the Lesbegue dominated convergence theorem,

$$\lim_{t\to 0} L_{h_t}(f) = \lim_{t\to 0}\int e^{-\frac{x^2 t}{2}} f(x)\ dx = \int f(x)\ dx. \tag{2.3.2}$$

On the other hand,

$$L_{h_t}(\widehat{f}) = \sqrt{2\pi} L_{p_t}(f) \longrightarrow \sqrt{2\pi} f(0) \tag{2.3.3}$$

by formula (4.1.6) and lemma 4.1 in L. Gross's notes.

The conclusion is that $T = \frac{1}{2\pi}$. Putting all of this together, we get the formula for Forier inversion.

3. OSCILLATOR REPRESENTATION

3.1. Recall $V = \mathcal{S}(\mathbb{R})$, rapidly decreasing functions. The Lie algebra

$$sl(2,\mathbb{C}) = \left\{ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

acts on V by the formulas:

$$\varpi(h) = x\partial_x + \frac{1}{2}$$

$$\varpi(e) = \frac{i}{2}x^2 \tag{3.1.1}$$

$$\varpi(f) = \frac{i}{2}\partial_x^2$$

For a linear space V, denote by End$V$, the space of linear maps $L : V \longrightarrow V$. This is again vector space, with the usual adition and scalar multiplication

$$(L_1 + L_2)(v) := L_1(v) + L_2(v), \qquad (\alpha L)(v) := \alpha L(v). \tag{3.1.2}$$

To say that a Lie algebra $\mathfrak{g}$ acts on a space V, means that there is a linear map

$$\varpi : \mathfrak{g} \longrightarrow \mathrm{End}(V) \tag{3.1.3}$$

which also satisfies

$$\varpi([x,y]) = \varpi(x)\varpi(y) - \varpi(y)\varpi(x). \tag{3.1.4}$$

We can also exponentiate the operators in (3.1.1):

(1) $\omega(e^{th})F(x) = e^{t/2}F(e^t x)$
(2) $\omega(e^{te})F(x) = e^{itx^2/2}F(x)$
(3) $\omega(e^{tf})F(x) =$ convolution with $(\frac{1+i}{2})(\pi t)^{-1/2}e^{-ix^2/2t}$
(1) and (2) are "easy". (3) needs Fourier transform.

The reason for the $\frac{i}{2}$ is that (1) and (2) are unitary operators w.r. to $L^2(\mathbb{R})$.

We are interested in the operator

$$2i\omega(e-f) = x^2 - \partial_x^2 \quad \overline{\mathbf{k}} := i(e-f).$$

This is called the Hermite operator. We want eigenfunctions.

If we write

$$a^- = x + \partial_x, \quad a^+ = x - \partial_x,$$

we get

$$[a^-, a^+] = 2, \ \frac{1}{4}(a^+a^- + a^-a^+) = \varpi(k).$$

This is also a Lie algebra representation. The Heisenberg algebra

$$\{p, q, z\}, \ [p, q] = z, \ [p, z] = 0, \ [q, z] = 0.$$

z acts by multiplication by 2.

$[a^-, (a^+)^j] = 2j(a^+)^{j-1}$

$$\begin{aligned} [a^-, a^{+2}] &= a^-a^{+2} - a^{+2}a^- = a^-a^{+2} - a^+(2 + a^-a^+) \\ &= a^-a^{+2} - 2a^+ - a^+a^-a^+ = a^-a^{+2} - (2 + a^-a^+)a^+ - 2a^+ = 4a^+. \end{aligned}$$

Let $v_0 = e^{-x^2/2}$. Then $a^-v_0 = 0$. Set $v_j := (a^+)^j v_0, \ j \in \mathbb{N}$. Then $v_j \in s(\mathbb{R})$. So $a^+v_j = v_{j+1}$. Furthermore

$$a^-v_j = a^-(a^+)^j v_0 = 2ja^{+j-1}v_0 = 2_j v_{j-1}$$

$$(v_j, v_\ell) = 2^\ell \ell! \delta_{j\ell}(v_0, v_0) = 2^\ell \ell! \sqrt{\pi} \delta_{j\ell}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Conclusion. $v_j = P_j(x)e^{-x^2/2}$, P_j a polynomial. P_j called Hermite Polynomial.

Theorem. *Hermite functions form an orthogonal basis for $L^2(\mathbb{R})$.*

$$\varpi(k)v_j = \big(j + \frac{1}{2}\big)v_j$$

4. REVIEW OF SOME LINEAR ALGEBRA

4.1. Let A be an $n \times n$ matrix with complex entries. The minimal polynomial is defined as the lowest degree polynomial $m = m_A m(t)$ such that $m(A) = 0$. The chracteristic polynomial, $p = p_A$, is defined as

$$p(t) = \det(tI - A). \tag{4.1.1}$$

Then

$$m(t) = \prod (t - \lambda_i)^{m_i}, \qquad p(t) = \prod (t - \lambda_i)^{n_i}. \tag{4.1.2}$$

The λ_i are called the (generalized) eigenvalues. The Cayley-Hamilton theorem implies that $1 \le m_i \le n_i$.

4.2. The results in this section hold over any field. Assume that $m = p \cdot q$, with p, q polynomials, such that $(p, q) = 1$. The Euclidean algorithm implies that there are polynomials a, b such that

$$ap + bq = 1. \tag{4.2.1}$$

Then

$$a(A)p(A) + b(A)q(A) = I. \tag{4.2.2}$$

Let

$$V_p := q(A)V, \qquad V_q = p(A)V. \tag{4.2.3}$$

The first conclusion is that

$$V = V_p + V_q, \qquad p(A)V_p = (0), \quad q(A)V_q = (0). \tag{4.2.4}$$

The second part is clear; for example $p(A)V_p = p(A)q(A)V = m(A)V = (0)$ by the definition of m. For the first part,

$$v = Iv = p(A)a(A)v + q(A)b(A)v. \tag{4.2.5}$$

In addition $V_p \cap V_q = (0)$; if $v = p(A)x = q(A)y$, then

$$\begin{aligned} v =& a(A)p(A)v + b(A)q(A)v = a(A)p(A)q(A)y + b(A)q(A)p(A)x = \\ =& a(A)y + b(A)m(A)x = 0 + 0 = 0. \end{aligned} \tag{4.2.6}$$

So if we change bases in such a way that say the first r vectors form a basis of V_p, and the last s vectors a basis of V_q, then the matrix A becomes block diagonal

$$A = \begin{pmatrix} A_p & 0 \\ 0 & A_q \end{pmatrix} \tag{4.2.7}$$

The minimal polynomial of A_p ois p, and the minimal polynomial of A_q is q. The similar result holds for when m decomposes into more than two factors, mutually prime to each other.

4.3. We return to the setting of section 4.1. There is a basis such that the matrix A is block diagonal,

$$A = \begin{pmatrix} A_{\lambda_1} & 0 & \dots & 0 \\ 0 & A_{\lambda_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & A_{\lambda_k} \end{pmatrix} \tag{4.3.1}$$

Each A_{λ_i} has minimal polynomial $(t - \lambda_i)^{m_i}$, and each block has size n_i. The spaces V_{λ_i} are called generalized λ_i eigenspaces.