Mathematics 401 **Spring 2006**

Topics in Analysis

This set of projects will deal with solutions of partial differential equations, Fourier series and their realtions to group representations.

Fundamental solutions. The Laplace operator on $\mathbb{R}^{p+q}$ is

$$\Delta_{p,q} = \sum_{i\le p} \frac{\partial}{\partial x_i} - \sum_{p<j} \frac{\partial}{\partial x_j}$$

and solutions to the equation $\Delta f = g$ (called the Poisson equation) are basic for many problems coming from applications. To solve this it is sufficient to solve

$$\Delta S = \delta_0$$

where δ_0 is the *Dirac delta function*. The siolution S is called a fundamental solution. The general equation is solved by

$$f = S * g,$$

where $*$ is convolution

$$S * g(x) = \int_{\mathbb{R}^n} S(x-y)g(y)\; dy, \qquad n = p+q.$$

For example if $q = 0$, the fundamental solution S is given by

$$S = \begin{cases} c_p(x_1^2 + \cdots + x_n^2)^{-p-2} \text{ if } p \neq 2, \\ c_2 log(x_1^2 + \cdots + x_n^2) \text{ if } p = 2. \end{cases} \tag{0.0.1}$$

There are several ways to get this result. An important feature is the invariance of the equation by the orthogonal groups.

Introductory material

(1) Fourier transform on the real line,
(2) distribution theory,
(3) the Γ function and its properties.

Projects

(1) generalized functions f^λ for $\lambda \in \mathbb{C}$,
(2) the Heisenberg group
(3) the Weyl algebra (PDE with polynomial coefficients)
(4) Stone-von Neumann theorem

D-modules and applications. The problem of finding a fundamental solution for a partial differential operator with polynomial coefficients can be solved algebraically by using the theory of D-modules also called modules of the Weyl algebra.

Introductory material

(1) the generalized functions r^λ,
(2) the orthogonal groups,
(3) modules over a ring,
(4) derivations, rings of differential operators,
(5) Lie algebras,
(6) Heisenberg Lie algebra,
(7) basic properties of the Weyl algebra,

Projects

(1) finite dimensional representations of $sl(2, \mathbb{R})$,
(2) spherical harmonics,
(3) fundamental solutions via D-modules,
(4) the Jacobian conjecture,
(5) stability of differential equations,
(6) automatic proof of identities,
(7) the metaplectic representation and harmonic analysis,

The fast Fourier transform.

Wavelets.

The Banach-Tarski paradox.

References

- Banach, S. and Tarski, A. "Sur la dcomposition des ensembles de points en parties respectivement congruentes." Fund. Math. 6, 244-277, 1924.
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The Runge-Kutta method and Hopf algebras.

Consider the ordinary differential equation $y' = f(y)$ with initial condition $y(x_0) = y_0$. Since most of the time it cannot be solved exactly, one often resorts to numerical solutions. One of the simplest methods is the Euler method. To estimate the value of y at x, one divides the interval $[x_0, x]$ into n subintervals with endpoints $x_0, x_1, \dots x_n = x$ and then estimates

$$y_i = y_{i-1} + (x_i - x_{i-1})f(y_{i-1}).$$

The various methods of estimation, Runge-Kutta being one of them, are elaborations on Euler's method. For estimating the error one needs estimates of the derivatives $y^{(n)}$. This can be done by differentiating the equation $y' = f(y)$. The simplest instance of this would be the following

Problem. *Let f and g be functions of one variable x. Find a formula for $y^{(n)}$ where $y = f(g(x))$.*

The first three derivatives are

$$y^{(1)} = f^{(1)}(g(x))g^{(1)}(x)$$

$$y^{(2)} = f^{(2)}(g(x))(g^{(1)}(x))^2 + f^{(1)}(g(x))g^{(2)}(x)$$

$$y^{(3)} = f^{(3)}(g(x))(g^{(1)}(x))^3 + 3f^{(2)}(g(x))g^{(1)}(x)g^{(2)}(x) + f^{(1)}(g(x))g^{(3)}(x).$$

The general answer was known for some time and is given in terms of *rooted trees*. Not surprisingly, they play a role in the error estimates for the more sophisticated methods. Perhaps more surprisingly, these combinatorics have found applications in other areas, such as Feynman diagrams.

Introductory material

(1) review of ODE and numerical methods

(2) Hopf algebras

Projects

(1) the Runge-Kutta method, its error estimates and the role of rooted trees
(2) the Hopf algebra attached to rooted trees and its application to other areas

References

[Boyce-DiPrima] *Ordinary differential equations*
[Hairer-Norsett-Wanner] *Solving ordinary differential equations I*
[Connes] *various expository and research papers by A. Connes and coworkers. Look in xArchiv*