A Transformation Groups Point of View on Connelly’s Global Rigidity Theorem for Free Equivariant Tensegrities

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1 Introduction

In the mid 1990’s, one of the authors [C4] generalized earlier work with Maria Terrell [CT] to prove a remarkable result on the global rigidity of certain symmetric tensegrities.

Global rigidity is a statement of transitivity for the action of the Euclidean motions on a certain sets of tensegrities. The purpose of this paper is to present a transformation groups and representation theoretic point of view for proving the main results of [C4].

The original proof already made essential use of the group algebra of the finite symmetry group G, and then related this to techniques previously developed in tensegrity theory. The appearance of the group algebra was one of the most surprising features of the original proof. The major feature of this paper is to formulate in an intrinsic way the group ring and representation theory related algebraic structures leading to the proof of Connelly’s theorem without dependence on other tensegrity results. However the basic approach of the proof is the same as the original. The intended contribution here (which took a quite a long period of effort to come up with) is to identify the relevant intrinsic and
natural structures on this class of tensegrities, and to avoid most dependence on general facts about tensegrities.

Several years ago the authors worked together using [C4] to produce a catalog containing a large subfamily of globally rigid tensegrities. Section 2.1 of this paper discusses the production of the catalog in a little more detail. The primary form of each example therein was an interactively viewable file (OOG format) using the mathematical visualization software geomview [GVW]. A static form of this catalog is at:


This paper may be regarded as another explanation of the mathematics behind that catalog.

A presentation of Simon Guest of some of his results in a talk helped us realize that the tensor product description given in section 2 was the natural functorial description of the set of tensegrities we had long been considering, and we’d like to express our appreciation to him for that.

2 Basic Definitions and the Main Result

A **tensegrity framework** may be viewed as a labeled graph \( \Gamma \) imbedded in a finite dimensional real inner product space \( W \). The imbedding is determined by a map \( \tau : V \to W \), \( V \) being the vertex set of \( \Gamma \), and the edges being mapped linearly between the vertices. The edges are labeled as **cables** or **struts**, these labels referring to distance constraints in comparisons to other tensegrities with the same labeled graph \( \Gamma \).

Specifically, let \( \tilde{W} \) be a finite dimensional inner product space containing \( W \). Another tensegrity \( \tilde{\tau} : \Gamma \to \tilde{W} \) with the same cable and strut structure is said to be **dominated** by \( \tau \) if the cables of \( \tilde{\tau} \) are no longer than those of \( \tau \) and the struts of \( \tilde{\tau} \) are no shorter than those of \( \tau \). Intuitively \( \tilde{\tau} \) is a candidate for the result of deforming \( \tau \) while respecting the cable and strut distance constraints. The tensegrity \( \tau \) is said to be **globally rigid** if any \( \tilde{\tau} \) dominated by \( \tau \) must be the image of \( \tau \) under an isometry of \( \tilde{W} \).

Suppose the finite group \( G \) acts orthogonally on \( W \) via a representation \( \rho \). A natural kind of tensegrity to consider is one which is invariant under the group action and for which the group action is transitive and effective on the set of vertices. We call this a (one orbit) free equivariant tensegrity. Upon picking one vertex \( v_0 \) as a basepoint, the cable and strut structure of the entire tensegrity is determined by listing two sets of group elements \( L_S = \{ h_1,..., h_s \} \) and \( L_C = \{ h_{s+1},..., h_{s+c} \} \). Here the vertices \( \rho(h_i)v_0 \) together with \( \rho(h_i^{-1})v_0 \) for \( 1 \leq i \leq s \) enumerate the other endpoints of all struts emanating from \( v_0 \). And the vertices \( \rho(h_i)v_0 \) together with \( \rho(h_i^{-1})v_0 \) for \( s + 1 \leq i \leq s + c \) enumerate the cables. We refer to the sets \( L_S \) and \( L_C \) respectively as strut and cable link sets.

We also require that none of the link set elements be the identity \( e \in G \).

We use the term **Cayley tensegrity of type** \((s, c)\) to refer to a tensegrity \( \tilde{\tau} \) with the same cable and strut structure as such a one orbit free equivariant
tensegrity, but with no assumption that the imbedding geometrically respects
the group action of \( G \). Thus the vertex set of such a tensegrity may be viewed
as \( G \) itself. Proposition 3.1 in Section 3 points out that there is strut joining
vertices \( q_1 \) and \( q_2 \) exactly when \( g_1^{-1}g_2 \) or \( g_2^{-1}g_1 \) is in the strut link set \( \mathcal{L}_G \). The
cable case is analogous.

Recall that the group algebra \( \mathcal{F}[G] \) is defined to be the set of functions
\( \phi : G \to \mathcal{F} \) where \( \mathcal{F} \) will be either the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \).
The constant function taking the value 1 is denoted \( 1_G \). We’ll see below that it plays
a natural role in describing translation. We follow Sternberg [S], Fulton-Harris
[FH], and Serre [Se] in our conventions for discussing group algebras.

Since a Cayley tensegrity of type \((s, c)\) is determined by the locations of the
vertices in the inner product vector space \( \tilde{W} \), we may view the tensegrity as an
element of \( \text{Map}(G, \tilde{W}) = \text{Map}(G, \mathbb{R} \otimes \tilde{W}) = \mathbb{R}[G] \otimes \tilde{W} \), the space of mappings
of the discrete set \( G \) into the vector space \( \tilde{W} \). \( \text{Map}(G, \tilde{W}) = G \times \tilde{W} \) is just
a vector space of dimension \( |G|d, |G| \) being the order of the group \( G \), and \( d \)
the dimension of \( \tilde{W} \). We denote by \( T_{s,c}(\tilde{W}; \Gamma) \) this space naturally isomorphic
to \( \mathbb{R}[G] \otimes \tilde{W} \) of all such Cayley tensegrities modelled on the labelled graph \( \Gamma \).
A simple product element \( \phi \otimes w \) would just correspond to a tensegrity whose
vertex corresponding to \( g \) were located at \( \phi(g)w \).

The affine transformations of \( \tilde{W} \) are mappings \( w \mapsto Aw + b \) where \( A \in
\text{End}(\tilde{W}) \) is linear and \( b \in \tilde{W} \) gives the translation part. We’ll denote this
semigroup as \( \text{Aff}_{\text{all}}(\tilde{W}) \). These affine transformations act on \( T_{s,c}(\tilde{W}; \Gamma) \) by
composition. The corresponding action of the affine transformation \( w \mapsto Aw + b \)
on \( \mathbb{R}[G] \otimes \tilde{W} \) is just via \( \Sigma \phi \otimes w \mapsto \Sigma \phi \otimes Aw + 1_G \otimes b \).

There is an augmentation map \( \epsilon : \mathbb{R}[g] \to \mathbb{R} \) defined by \( \phi \mapsto \Sigma_{g \in G} \phi(g) \),
and we denote its kernel by \( \mathbb{R}^0[g] \). Similarly we denote the kernel of \( \epsilon \otimes \text{Id} : T_{s,c}(\tilde{W}; \Gamma) \to \tilde{W} \) by \( T_{s,c}^0(\tilde{W}; \Gamma) \). For every tensegrity \( \tau \), there is a unique translation of \( \tau \) to an element in \( T_{s,c}^0(\tilde{W}; \Gamma) \). Geometrically, this translated tensegrity
is one whose centroid (average of the vertices) is 0.

The group algebra supports a two sided \( G \times G \) group action taking a function
\( f \) to the function \( \tilde{f} = (k_1, k_2)f \) defined by \( \tilde{f}(g) = f(k_1^{-1}gk_2) \). This action
satisfies the composition rule:

\[
(k_1, k_2) ((k_1, k_2)) f = (k_1k_1, k_2k_2) f.
\]

Restricting to the element \((k, e)\), we refer to the left \( G \) action \( L_k : \mathcal{F}[G] \to
\mathcal{F}[G] \). Similarly restricting to the element \((e, k^{-1})\), gives the right \( G \) action \( R_k : \mathcal{F}[G] \to \mathcal{F}[G] \). There is also a natural inner product \( \langle f, h \rangle = \Sigma_{g \in G} f(g)h(g) \)
on \( \mathbb{R}[G] \). Combined with the inner product structure of \( \tilde{W} \), this gives an inner
product structure on \( T_{s,c}(\tilde{W}; \Gamma) \).

Connelly’s theorem involves a nice construction using a family of quadratic
forms (energy forms) on the linear space \( T_{s,c}(\tilde{W}; \Gamma) \). These are based on what
we call Cayley spring constants, namely \( s + c \) numbers \( \omega = (\omega_1, \omega_2, \ldots, \omega_{s+c}) \)
with the strut constants (for \( 1 \leq i \leq s \)) satisfying \( \omega_i \leq 0 \) and the cable constants
(for \( s + 1 \leq i \leq s + c \)) satisfying \( \omega_i \geq 0 \). This energy form is defined by

\[
E(\tau; \omega, \hat{W}) = \sum_{g \in G} \sum_{i=1}^{s+c} m_i \omega_i \left\| \tau(g) - \tau(gh_i) \right\|^2
\]

where the multiplicity \( m_i \) of each cable or strut associated to the link element \( h_i \) is defined to be 2 if \( h_i \) has order greater than 2 and 1 if \( h_i \) has order 2. This sum is simply the weighted sum of half the lengths squared of all the edges, the weight of an edge being the Cayley spring constant corresponding to that cable or strut.

It is immediate from the definition that the energy is unchanged by translation of the tensegrity. So \( \sum_{g \in G} b \in \mathcal{R}^0[G] \otimes \hat{W} \) is orthogonal to all \( \tau \in T_{s,c}(\hat{W}; \Gamma) \).

And the signs of the spring constants are such that if \( \tau \) is dominated by \( \tau \), then \( E(\tau; \omega, \hat{W}) \leq E(\tau; \omega, W) \).

Proposition 3.2 in section 3 shows that the self adjoint linear transformation of \( T_{s,c}(\hat{W}; \Gamma) \) associated to this energy form is \( R \Lambda \otimes \text{Id} \) where \( R \Lambda \) is right multiplication by the element \( \Lambda = \sum_{i=1}^{s+c} m_i \omega_i (2 - (h_i + h_i^{-1})) \) of \( \mathcal{R}^0[G] \subset \mathcal{R}[G] \).

Globally rigid tensegrities are constructed by what we refer to here as the minimal self-stress construction. It involves considering a one parameter family of energy forms. Specifically,

**Minimal Self-Stress Construction**

- Start with the labeled graph \( \Gamma \) of an abstract Cayley tensegrity of type \((s, c)\) whose cable link set \( \mathcal{L}_C \) generates the group \( G \).
- Fix a \( c \)-tuple of positive cable stresses \( \omega_C = (\omega_{s+1}, \ldots, \omega_{s+c}) \) and an \( s \)-tuple \( \hat{\omega}_S = (\hat{\omega}_1, \ldots, \hat{\omega}_s) \) which is not identically zero and each entry of which is negative.
- For any value of the parameter \( \sigma \geq 0 \), let \( E_\sigma(\tau) \) be the energy form \( E(\tau; \omega, \mathcal{R}) \) on \( \mathcal{R}^0[G] \) where \( \omega = (\omega_S, \omega_C) \in \mathcal{R}^s \times \mathcal{R}^c \) and \( \omega_S = \sigma \hat{\omega}_S \).
- Define the real number \( \lambda \) by

\[
\lambda = \inf \{ \sigma > 0 : E_\sigma \text{ is not positive definite on } \mathcal{R}^0[G]. \}
\]

- The output is the minimal self stress \( \omega_\lambda = (\lambda \hat{\omega}_S, \omega_C) \) and the left \( G \) invariant subspace \( V_\lambda \subset \mathcal{R}^0[G] \) which is the nullspace of the semidefinite form \( E_\lambda \).

**Theorem 2.1** Given minimal self-stress initial data, the minimal self-stress construction above is a well defined construction which produces a left \( G \) invariant subspace \( V_\lambda \subset \mathcal{R}^0[G] \) which is the nullspace of the semidefinite form \( E_\lambda \).

**Proof:** Proposition 3.3 below establishes the positive definiteness of \( E_\sigma \) when \( \sigma = 0 \) (using the hypothesis that the cable link set generates the group.) Since
the strut stress vector is negative and not identically zero, it is obvious for \( \sigma \) large enough that the quadratic form \( E_{\sigma} \) will take on negative values for some tensegrities. Hence the number \( \lambda \) in the construction clearly exists, and by continuity, \( E_\lambda \) is positive semidefinite. The linear transformation associated to \( E_\sigma \) is right multiplication by the element \( \Lambda \in \mathbb{R}^0[G] \). Since \( E_\lambda \) is semidefinite, the zero set of the energy is a subspace and coincides with the kernel of the associated linear transformation. Right multiplication commutes with left multiplication by elements of \( G \), so the nullspace of this linear transformation is a left \( G \) (or \( \mathbb{R}[G] \)) invariant module.

We call \( \lambda \) the minimal self-stress multiple and \( V_\lambda \) the minimally stressed representation. We also refer to the c-tuple \( \omega_C \) and the s-tuple \( \hat{\omega}_S \) above respectively as cable and strut minimal self-stress initial data.

Suppose \( W \) is any inner product space. Fix minimal self-stress initial data and let \( \lambda \) be the associated minimal self-stress multiple and \( V_\lambda \) the minimally stressed representation. By tensoring with the inner product on \( W \), extend the energy form \( E_\lambda \) on \( \mathbb{R}[G] \) to the energy form \( E_{\lambda,W} \) on \( \mathbb{R}[G] \otimes W = T_{s,c}(W; \Gamma) \); this is just the energy form \( E(\tau; \omega_\lambda, W) \) discussed earlier.

**Theorem 2.2** Given minimal self-stress initial data, let \( \lambda \) be the minimal self-stress multiple and \( V_\lambda \) the minimally stressed representation.

(a) The nullspace of \( E_{\lambda,W} \) restricted to \( T_{s,c}^0(W; \Gamma) \) (the tensegrities in \( W \) with centroid 0) is naturally isomorphic to \( V_\lambda \otimes W \).

(b) On \( T_{s,c}(W; \Gamma) \), the nullspace is \( (V_\lambda \oplus \mathbb{R}) \otimes W \) where the \( \mathbb{R} \) factor consists of all multiples of \( 1_G \).

(c) If \( \tau \) is a zero energy tensegrity with respect to \( E_{\lambda,W} \) then any tensegrity \( \tilde{\tau} \) dominated by \( \tau \) is also in \( (V_\lambda \oplus \mathbb{R}) \otimes W \).

**Proof:** The quadratic form \( E_{\lambda,W} \) is positive semidefinite, so the energy 0 elements are the same as elements of the nullspace of the associated self adjoint linear transformation. The decomposition \( (T_{s,c}^0(W; \Gamma) \oplus \mathbb{R}) \otimes W = T_{s,c}(W; \Gamma) \) is an orthogonal one with respect to \( E_{\lambda,W} \) and \( E_{\lambda,W} \) restricts to zero on the \( \mathbb{R} \otimes W \) summand. The self-adjoint transformation associated to \( E_{\lambda,W} \) is \( R_\Lambda \otimes \text{Id} \) and the nullspace of \( E_\lambda \) restricted to \( \mathbb{R}^0[G] \) is just \( V_\lambda \). So the kernel of \( R_\Lambda \otimes \text{Id} \) acting on \( \mathbb{R}^0[G] \otimes W \) is just \( V_\lambda \otimes W \).

If \( \tilde{\tau} \) is dominated by \( \tau \), then \( E_{\lambda,W}(\tilde{\tau}) \leq E_{\lambda,W}(\tau) = 0 \). But \( E_{\lambda,W} \) positive semidefinite implies \( E_{\lambda,W}(\tilde{\tau}) = 0 \), and so \( \tilde{\tau} \) is in the nullspace above.

Since it preserves energy, the action of \( \text{Aff}_{s,c}(W; \Gamma) \) on the space of tensegrities \( T_{s,c}(W; \Gamma) \) restricts to an action on the 0 energy subspace \( (V_\lambda \oplus \mathbb{R}) \otimes W \). Explicitly the affine transformation \( u \mapsto Au + b \) takes \( \Sigma \phi \otimes w \) to \( \Sigma \phi \otimes Aw + 1_G \otimes b \). Here \( V_\lambda \oplus \mathbb{R} \) is a subspace of \( \mathbb{R}[G] \).
The tensor product \((V_\lambda \oplus \mathcal{R}) \otimes W\) is canonically identified with the space of linear maps \(\text{Lin}(\mathcal{R})^*, W)\). And the action of \(\text{End}(W)\) on this space of zero energy tensegrities just corresponds to the composition action of \(\text{End}(W)\) on the space of linear maps.

**Theorem 2.3** Given minimal self-stress initial data, let \(\lambda\) be the minimal self-stress multiple and \(V_\lambda\) the minimally stressed representation. Suppose \(\dim(W) \geq \dim(V_\lambda)\) and \(\tau\) is a maximal rank (viewed as an endomorphism) element of the zero energy tensegrities \(V_\lambda \otimes W \subset T_{s,c}^0(W; \Gamma)\) with centroid 0. Then any other tensegrity \(\tilde{\tau} \in T_{s,c}^0(W; \Gamma)\) dominated by \(\tau\) is the image of \(\tau\) under an element \(w \mapsto Aw + b\) of \(\text{Aff}(W)\). And the affine transformation \(w \mapsto Aw + b\) must preserve the edge lengths of the tensegrities.

**Proof:** By 2.2 above, we know \(\tilde{\tau} \in (V_\lambda \oplus \mathcal{R}) \otimes W\). By composing with a translation we can shift the centroid of \(\tilde{\tau}\) to the origin so that \(\tilde{\tau} \in V_\lambda \otimes W \subset T_{s,c}^0(W; \Gamma)\). The observation that the composition action of \(\text{End}(W)\) on \(\text{Lin}(V_\lambda^*, W)\) is transitive (under the assumed dimension inequality) completes the proof that \(\tilde{\tau}\) is an affine image of \(\tau\). And the (strict) negativity of the Cayley spring constants for the struts combined with the (strict) positivity for cables together with both tensegrities being in the nullspace of the energy shows that the edge lengths must match.

Suppose now that the vector space \(W\) is acted upon orthogonally by \(G\) via a linear representation \(\rho\). (The case \(W = V_\lambda\) is an example of this.) There are several natural maps which now enter:

a) Given \(\phi \in \mathcal{R}[G]\), its Fourier transform \(\hat{\phi}\) associates to the representation space \(W\) the endomorphism of \(W\) defined by \(\hat{\phi}(\rho)(w) = \sum_{g \in G} \phi(g)(\rho(g))(w)\).

b) Evaluation \(e : \mathcal{R}[G] \otimes W \to W\) defined by \(e(\phi \otimes w) = \hat{\phi}(\rho)w\). This is just the scalar multiplication map for the \(\mathcal{R}[G]\) module structure of the \(G\) space \(W\).

c) Equivariant extension \(i : W \to \mathcal{R}[G] \otimes W\) defined by \(w \mapsto \sum_{g \in G} \delta_g \otimes \rho(g)w\) where \(\delta_g \in \mathcal{R}[G]\) is the function satisfying \(\delta_g(h) = 1\) and \(\delta_g(h) = 0\) for \(h \neq g\). The map \(i\) takes a point \(v\) to the equivariant tensegrity whose vertex corresponding to the identity is \(v\).

d) The inversion map \(\text{Inv} : \mathcal{R}[G] \to \mathcal{R}[G]\) defined by \(\text{Inv}(\phi)(g) = \phi(g^{-1})\). The identity \((g_1g_2)^{-1} = g_2^{-1}g_1^{-1}\) means this map interchanges left and right multiplication on the group algebras.

Recall \(\Lambda = \sum_{i=1}^{\ell} \alpha_i \omega_i (2 - (h_i + h_i^{-1})) \in \mathcal{R}[G]\).

**Theorem 2.4** Let \(\lambda\) be the minimal self-stress multiple, \(V_\lambda\) the minimally stressed representation, and \(\rho\) an orthogonal representation of \(G\) on \(W\).
a) If $\tau \in T_{s,c}(W; \Gamma)$ is a zero energy tensegrity, then the element $e(\text{Inv}(\tau) \otimes \text{Id})$ of $W$ is in the kernel of the endomorphism $\rho(\Lambda)$.

b) Conversely if $w_0 \in W$ is a nonzero element satisfying $\rho(\Lambda)w_0 = 0$, then its equivariant extension $i(w_0)$ is a nontrivial zero energy element $\tau \in T_{s,c}(W; \Gamma)$. And if the decomposition into irreducibles of the representation $\rho$ does not contain a trivial one dimensional representation as a summand, then the equivariant extension $i(w_0)$ has centroid 0; namely $i(w_0) \in T^0_{s,c}(W; \Gamma)$.

**Proof:**

a) Since $\tau$ is in the nullspace of $E_{\lambda,W}$, by proposition 3.2, 

$$(R_{\Lambda} \otimes \text{Id}) \tau = 0.$$ 

Applying $e \circ (\text{Inv} \otimes \text{Id}) : \mathcal{R}[G] \otimes V \to V$ gives us by Propositions 3.5 and 3.4

$$0 = e \circ (\text{Inv} \otimes \text{Id}) \circ (R_{\Lambda} \otimes \text{Id}) \tau = e \circ (L_{\Lambda} \otimes \text{Id}) \circ (\text{Inv} \otimes \text{Id}) \tau = \rho(\Lambda) \circ e \circ (\text{Inv} \otimes \text{Id}) \tau.$$ 

b) By proposition 3.6,

$$\rho(\Lambda)w_0 = 0 \Rightarrow ((R_{\Lambda} \otimes \text{Id}) \circ i)(w_0) = (i \circ \rho(\Lambda))w_0 = 0.$$ 

So the tensegrity $i(w_0)$ given by equivariant extension is a zero energy tensegrity. Its vertex at the identity of $G$ is $w_0$, so it is a nonzero tensegrity. Its centroid is given by 

$$\Sigma_{g \in G} \rho(g)w_0$$ 

which for all $g$ is a $\rho(g)$ invariant element of $W$. So as long as $\rho$ does not have a trivial representation as a direct summand, this vector must be zero and the centroid of $i(w_0)$ is at the origin.

As described in e.g. chapter 6 of [Se], the complex group algebra decomposes as a product 

$$\mathbb{C}[G] = \Pi_{\alpha \in A} \text{End}_CE_\alpha$$ 

where the product is over the set of distinct complex irreducible representations $\rho_\alpha$ of $G$. We can take each $\rho_\alpha$ to be a unitary representation of $G$ on the complex vector space $E_\alpha$. This identification is obtained by using the Fourier transform as described earlier to associate to each $\phi \in \mathbb{C}[G]$ the direct product of endomorphisms $\Pi_{\alpha \in A} \phi(\rho_\alpha)$.

Further, as described e.g. in chapter 13 of [Se], the real group algebra $\mathbb{R}[G]$ which enters into our description of the space $T_{s,c}(W; \Gamma)$ also factors into a
product of matrix algebras, but there are three different possibilities. These
are indexed by the three kinds of irreducible unitary representations of a finite
group given below.

**Real** The complex representation $\rho_\alpha$ has an invariant symmetric bilinear
form. $E_\alpha = W_\alpha \otimes_R \mathbb{C}$ for a real vector space $W_\alpha$ on which $G$ acts orthogonally. The corresponding matrix algebra factor of $\mathcal{R}[G]$ is $\text{End}_R(W_\alpha)$. The inclusion $\mathcal{R}[G] \to \mathcal{C}[G]$ corresponds on this factor to the map $\text{End}_R(W_\alpha) \to \text{End}_C(E_\alpha)$.

**Quaternionic** The complex representation $\rho_\alpha$ has an invariant skew symmetric bilinear form, which gives the even dimensional complex vector space $E_\alpha$ a skew symmetric conjugate linear map $J$ satisfying $J^2 = -1$. If we let $W_\alpha$ denote the real vector space obtained from $E_\alpha$ by forgetting its complex structure, then the corresponding matrix algebra factor of $\mathcal{R}[G]$ is $\text{End}_H(W_\alpha)$; i.e. the real linear endomorphisms commuting with both $J$ and the real endomorphism corresponding to multiplication by $i$. The inclusion $\mathcal{R}[G] \to \mathcal{C}[G]$ corresponds on this factor to the map $\text{End}_H(W_\alpha) \to \text{End}_C(E_\alpha)$.

**General Complex** In this case the representations $\rho_\alpha$ and $\bar{\rho}_\alpha$ are different and the representation has no nontrivial invariant bilinear forms. Once again let $W_\alpha$ denote the real vector space obtained from $E_\alpha$ by forgetting its complex structure. The corresponding matrix algebra factor of $\mathcal{R}[G]$ is $\text{End}_C(W_\alpha)$; i.e. the real linear endomorphisms commuting with the real endomorphism corresponding to multiplication by $i$. The inclusion $\mathcal{R}[G] \to \mathcal{C}[G]$ corresponds on this factor to the map $\text{End}_H(W_\alpha) \to \text{End}_C(E_\alpha) \times \text{End}_C(\bar{E}_\alpha)$.

Thus we have

$$\mathcal{R}[G] = \Pi_{\beta \in B} \text{End}_{F_{\beta}}(W_{\beta})$$

where $F_{\beta}$ can be any of the three possibilities $\mathcal{R}$, $\mathcal{C}$, or $\mathcal{H}$. And the set $B$ may be thought of as the set of irreducible real representations of $G$. Equivalently $B$ is the set of complex irreducible representations representations after the identification of $\rho_\alpha$ with $\bar{\rho}_\alpha$ in the general complex case since they both correspond to the same irreducible real representation. Under the product decomposition of the group algebra into spaces of endomorphisms factors, the maps $L_g$, $R_g$, and $\text{Inv}$ correspond on the relevant subspace $\text{End}_F(W_{\alpha}) \subset W_\alpha \otimes W_\alpha^*$ to $\rho(g)$, $\rho(g)^*$, and transpose respectively. More details on this are in proposition 3.7.

With respect to the product decomposition of $\mathcal{C}[G]$ above, the action of $\mathcal{C}[G]$ on one of the complex irreducible representations $E_\alpha$ is trivial for each $\text{End}_C(E_\alpha)$ factor with $\gamma \neq \alpha$. One way to see this is using the inverse Fourier transform in the complex case. (See e.g. chapter 6 of [Se].) It says that the element of $\phi \in \mathcal{C}[G]$ corresponding to an element $(u_1, \ldots, u_{|A|}) \in \Pi_{\alpha \in A} \text{End}_C(E_\alpha)$ is given by

$$\phi(g) = \frac{1}{|G|} \sum_{\alpha \in A} \text{dim}(E_\alpha) \text{Tr}(\rho_\alpha(g^{-1})u_\alpha).$$
Specializing this to an element of the product with only one factor $u_\alpha$ nontrivial, we see that the action of $\phi$ on a different $E_\gamma$ is zero by virtue of orthogonality relations among the matrix elements of the complex irreducible representations. The corresponding statement for the product decomposition of $R[G]$ acting on an element of the product with only one factor $u_\beta$ nontrivial follows from the complex case.

Theorem 2.5 Given minimal self-stress initial data, let $\lambda$ be the minimal self-stress multiple, and $V_\lambda$ the minimally stressed representation. Then for at least one of the matrix algebra factors $\text{End}_{F_{\gamma}}(W_\gamma)$ of $R[G]$ (where $F_\gamma = R$, $C$, or $H$), $\rho_\gamma(\Lambda)$ is a singular endomorphism of $W_\beta$, where $\rho_\gamma$ is the representation of $G$ on $W_\gamma$.

Moreover, suppose the kernel of $\rho_\gamma(\Lambda)$ acting on such a $W_\gamma$ is one dimensional with $w_0$ a nonzero element. Let $\tau_\beta = i(w_0) \in R[G] \otimes W_\beta$ be the associated equivariant extension. Then

a) The element $\tau_\beta$ is a maximal rank element of $V_\lambda \otimes W_\beta$.

b) $V_\lambda \cap \text{End}_{F_{\gamma}}(W_\beta)$ is equivariantly isomorphic to $W_\beta^*$ where $\text{End}_{F_{\gamma}}(W_\beta)$ refers to the appropriate factor as a subset of $R[G]$.

Proof: Since $\lambda$ is the minimal self-stress multiple, right multiplication by $\Lambda$ on $R[G]$ has a nontrivial kernel. Hence on one of the $\text{End}_{F_{\gamma}}(W_\gamma)$ factors, $R_\lambda$ has a nontrivial kernel. If $\phi \in \text{End}_{F_{\gamma}}(W_\gamma) \subset R[G]$ is a nonzero element of this kernel, then by proposition 3.7, $\phi(\rho_\gamma) \circ \rho_\gamma(\Lambda) = 0$. (This uses the fact that $\text{Inv}(\Lambda) = \Lambda$.) Thus $\phi(\rho_\gamma)$ is a nonzero element of $\text{End}_{F_{\gamma}}(W_\beta)$ whose composition with the element $\rho_\gamma(\Lambda)$ of $\text{End}_{F_{\gamma}}(W_\beta)$ is identically zero. Applying transposes, we see that $\rho_\gamma(\Lambda)$ has a nontrivial element $w_0 \in W_\beta$ in its kernel.

Under the one dimensional kernel assumption

a) Let $\pi_\beta$ be the projection $R[G] \rightarrow \text{End}_{F_{\gamma}}(W_\beta)$. Note $\tau_\beta$ is a nonzero tenseg

tegrity. By proposition 3.9

$$(e \circ (\text{Inv} \otimes \text{Id})) (\tau_\beta) = (e \circ (\text{Inv} \otimes \text{Id}) \circ i) w_0 = |G|w_0 \neq 0.$$ 

where $e$ and $i$ refer to evaluation and equivariant extension using $W_\beta$. But as discussed in the paragraph above, this evaluation map $e$ is zero on factors $\text{End}_{F_{\gamma}}(W_\gamma) \subset R[G]$ with $\gamma \neq \beta$. Consequently $(e \circ (\text{Inv} \otimes \text{Id})) (\tau_\beta) \neq 0$ implies $\pi_\beta(\tau_\beta) \neq 0$.

Part b) of theorem 2.4 shows that both $\tau_\beta$ and $\pi_\beta(\tau_\beta)$ lie in $V_\lambda \otimes W_\beta$.

Proposition 3.8 says $(L_k \otimes \text{Id})\tau_\beta = (\text{Id} \otimes \rho(k^{-1}))\tau_\beta$ which may be interpreted as a $G$ equivariance statement about the nonzero linear map $W_\beta^* \rightarrow V_\lambda \cap \text{End}_{F_{\gamma}}(W_\beta)$ associated to $\pi_\beta(\tau_\beta)$. The Schur lemma then
guarantees that this map is injective, and so $\tau_\beta$ is a maximal rank element of $V_\lambda \otimes W_\beta$.

b) $V_\lambda \cap \text{End}_{F_\beta}(W_\beta)$ is the kernel of $R_\Lambda$ on $\text{End}_{F_\beta}(W_\beta) \subset R[G]$. By proposition 3.7, multiplication by $R_\Lambda$ corresponds to precomposition with $\rho(\Lambda)$ on this factor. The kernel of $\rho(\Lambda)$ being only one dimensional then guarantees that $V_\lambda \cap \text{End}_{F_\beta}(W_\beta)$ is irreducible and the map $W_\beta^* \to V_\lambda \cap \text{End}_{F_\beta}(W_\beta)$ given by $\pi_\beta(\tau_\beta)$ is an isomorphism.

We wish now to combine theorems 2.3 and 2.5. For an orthogonal real representation (such as $W_\beta$ above), the inner product gives an equivariant identification between the vector space and its dual. So in considering vector spaces $W$ with $\dim(W) \geq \dim(V_\lambda)$ in which to seek maximal rank elements $\tau \in V_\lambda \otimes W$ to use in theorem 2.3, the smallest dimensional possibility is $W = V_\lambda$ itself.

The minimal self-stress construction always produces a minimally stressed representation $V_\lambda$ and a minimal self-stress multiple $\lambda$ for which $R_\Lambda$ is a singular endomorphism of $R[G]$. By theorem 2.5 above, $\rho_\beta(\Lambda)$ will be singular for at least one real irreducible representation $W_\beta$. Often (e.g. based on our web catalog) there will only be one such irreducible representation, the kernel will be one dimensional, and by the latter part of theorem 2.5, the equivariant extension $\tau_\beta = i(w_0) \in R[G] \otimes W_\beta$

will automatically be a maximal rank element of $V_\lambda \otimes W_\beta \simeq V_\lambda \otimes V_\lambda$. Regardless of how the maximal rank nature of $\tau \in V_\lambda \otimes V_\lambda$ was established, by theorem 2.3, as long as $\dim(\tilde{W}) \geq \dim(V_\lambda)$

any other tensegrity $\tilde{\tau} \in T_{s,c}(\tilde{W}; \Gamma)$ dominated by such a maximal rank tensegrity $\tau$ is the image of $\tau$ under an affine edge length preserving transformation $w \mapsto Aw + b$.

In this case, a standard rigidity definition and easy consequence re-expresses this. A tensegrity in $W$ is said to lie on a conic at infinity if there is a nontrivial quadratic form on $W$ which vanishes on every edge of the tensegrity. For Cayley tensegrities $\tau$ and $\tilde{\tau}$ of type $(s,c)$ related by an affine edge length preserving transformation $w \mapsto Aw + b$, we see for each element $h_i$ of one of the link sets, that

$$\|A\tau(gh_i) - A\tau(g)\|_2^2 = \|\tau(gh_i) - \tau(g)\|_2^2.$$ 

In other words the quadratic form $A^tA - I$ vanishes on the edges of $\tau$. The additional requirement for global rigidity of $\tau$ beyond the affine image result already established is that the linear map $A$ be forced to be distance preserving on its domain. Hence we have

**Theorem 2.6** A maximal rank equivariant tensegrity $\tau \in V_\lambda \otimes V_\lambda$ (such as one produced in part a) of the latter portion of theorem 2.5) is globally rigid if it does not lie on a conic at infinity.
2.1 Computer Approach for the Web Catalog

The minimal self-stress construction can be carried out computationally as follows:

- Enumerate the link sets \( \{h_1, ..., h_{s+c}\} \) of interest.
- Enumerate the real orthogonal irreducible representations \( \rho_\beta \) of \( G \).
- Choose cable and strut minimal self-struss initial data.
- For each real orthogonal irreducible representation, find the minimal \( \lambda \) so that \( \rho_\beta(\Lambda) \) is singular. The smallest of these among all \( \beta \) is the minimal self-stress multiple.
- If the kernel of \( \rho_\beta(\Lambda) \) is one dimensional so that part a) of Theorem 2.5 applies, use the equivariant extension \( \tau_\beta = i(w_0) \) to produce the candidate globally rigid tensegrity.

In our actual implementation, for reasons of practical efficiency minimal \( \lambda \) values were determined by numerical polynomial solution routines (with tolerancing and carried out in Maple) supplemented by Sturm sequence routines to numerically confirm that we hadn’t overlooked any roots.

No explicit tests were made to check for the conic at infinity possibility. Nor were examples with overlapping vertices immediately removed when they arose.

The computer catalog attempted to enumerate and visualize (primarily with 3d geomview [GVW] models, secondarily with static images) globally rigid Cayley tensegrities of type (1,2) (1 strut, 2 cables) subject to the following conditions:

- They were products of the Minimal Self-Stress Construction.
- They were associated with irreducible orthogonal three dimensional representations and so the symmetry groups were one of the six given in the table below. Thus they fall in the case described in the last part of theorem 2.5 where maximal rank is automatic for the equivariant extension \( i(w_0) \).
- The initial 2-tuple of cable stresses was \((1,1)\) and \(\hat{w}_S = (1)\).

Link sets \((h_1, h_2, h_3)\) conjugate by an inner automorphism of \( G \) (or obtained by interchanging \( h_1 \) and \( h_2 \)) produce isometric examples, so Maple was first used to enumerate the equivalence classes of pairs \((h_1, \{h_2, h_3\})\) under this action. The table below indicates how many equivalence classes \( \{h_2, h_3\} \) there are (which also generate the group \( G \)) for each of the six groups of interest.
Detailed information about the examples in the web catalog lie on web pages such as A4/html/Graph1/tree_html1.1.3.html. (Less complete information lies on the corresponding page short_tree_html1.1.3.html. The internal web links in the catalog generally only point to the short_tree form.) Here A4 can be replaced by any of the six groups. The number r following Graph in the above path runs from 1 to the number m of distinct cable pair equivalence classes as given in the above table. Examples with these cable pairs are indexed by an integer s running from 1 to some number p; thus the page Graphr/tree_htmlr1.r.p.html summarizes the results of the Minimal Self-Stress Construction for all link set equivalence classes with a particular cable link set equivalence class. For each example, phrases like

"Smallest positive root is .666666 occurring in representation dim3."

are just reporting .666666 as the minimal self-stress multiple λ and dim3 as the minimally stressed representation \( V_\lambda \). A "near-null vector" is also reported - this is the location of the vertex \( \tau(e) \) for the associated tensegrity. Pictures are only generated for three dimensional examples.

### 3 Technical Details

**Proposition 3.1** Let \( \Gamma \) be a Cayley tensegrity diagram of type \((s,c)\) with strut link set \( \mathcal{L}_S \) and cable link set \( \mathcal{L}_C \). Then:

- There is a strut \([g_1, g_2]\) joining vertices \( g_1 \) and \( g_2 \) iff \( g_1^{-1}g_2 \in \mathcal{L}_S \) or \( g_2^{-1}g_1 \in \mathcal{L}_S \).
- There is a cable \([g_1, g_2]\) joining vertices \( g_1 \) and \( g_2 \) iff \( g_1^{-1}g_2 \in \mathcal{L}_C \) or \( g_2^{-1}g_1 \in \mathcal{L}_C \).

**Proof:** There is an edge between \( g_1 \) and \( g_2 \) iff this edge is the image by an element \( g \in G \) of an edge from the identity \( e \) to some other vertex \( g_3 \). If the edge is a strut, then \( g_3 \in \mathcal{L}_S \) or \( g_3^{-1} \in \mathcal{L}_S \). Denote this as \( g_3^\epsilon \) where \( \epsilon \in \{\pm 1\} \). Since the vertex set is just \( G \) with a \( G \) action by left multiplication, it is clear that the element \( g \) must be \( g_1 \) or \( g_2 \). If \( g = g_1 \), then \( g \) must take the vertex \( g_3^\epsilon \) to the vertex \( g_2 \). So \( g_2 = g_1g_3^\epsilon \) and \( g_3^\epsilon = g_1^{-1}g_2 \). In the strut case this gives \( g_1^{-1}g_2 \in \mathcal{L}_S \) or \((g_1^{-1}g_2)^\epsilon \in \mathcal{L}_S \) as was to be shown.

If \( g = g_2 \), the same argument leads to the same conclusion. And the cable case is completely analogous.
Proposition 3.2 For a Cayley tensegrity of type \((s, c)\), the self adjoint linear transformation of the space of tensegrities \(T_{s,c}(\mathbb{W}; \Gamma) = \mathcal{R}[G] \otimes \mathbb{W}\) associated to the symmetric quadratic form \(E(\tau; \omega)\) is

\[
\sum_{i=1}^{s+c} m_i \omega_i (2 - (R_{h_i} + R_{h_i^{-1}})) \otimes \text{Id}.
\]

(Note here that \((R_h \otimes \text{Id})\tau)(g) = \tau(gh).)

Proof: Via polarization, the bilinear form \(B\) associated to \(E(\tau; \omega)\) is given by

\[
B(\tau_1, \tau_2) = \sum_{i=1}^{s+c} m_i \omega_i (\langle \tau_1(g), \tau_2(g) \rangle + \langle \tau_1(gh_i), \tau_2(gh_i) \rangle - \langle \tau_1(g), \tau_2(gh_i) \rangle - \langle \tau_1(gh_i), \tau_2(g) \rangle)
\]

In the usual rigidity terminology, the endomorphism

\[
\Omega = \sum_{i=1}^{s+c} m_i \omega_i \left(2 - (R_{h_i} + R_{h_i^{-1}})\right)
\]

of \(\mathcal{R}[G]\) which appears above is called the stress matrix. We will also view this as right multiplication by the element \(\Lambda = \sum_{i=1}^{s+c} m_i \omega_i \left(2 - \omega_i(h_i + h_i^{-1})\right)\) of \(\mathcal{R}[G]\).

Proposition 3.3 Suppose the cable link set \(\mathcal{L}_C\) generates the group \(G\). Let \(\omega_C = (\omega_{s+1}, \ldots, \omega_{s+c})\) be a \(c\)-tuple of positive numbers and \(\omega_S\) the 0 vector in \(\mathbb{R}^s\). Then with this choice \(\omega = (\omega_S, \omega_C)\) is \(\tau \in \mathbb{R}^s \times \mathbb{R}^s\) of spring constants, the energy form \(E(\tau; \omega)\) on \(T^0_{s,c}(\mathbb{W}; \Gamma)\) is positive definite.

Proof: All the coefficients are nonnegatative so clearly

\[
E(\tau; \omega) = \sum_{g \in G} \sum_{i=s+1}^{s+c} m_i \omega_i |\tau(g) - \tau(gh_i)|^2 \geq 0
\]

and \(E\) is positive semidefinite.

To establish definiteness, suppose \(E(\tau; \omega) = 0\). Then for each \(g \in G\) and for each \(h_i \in \mathcal{L}_C\)

\[
\tau(g) = \tau(gh_i).
\]
Since the cable link set generates $G$, every element $k \in G$ can be expressed as a product $e_0 h_{i_1} h_{i_2} \ldots h_{i_m}$ where $e_0$ is the identity in $G$. So the equalities $\tau(g) = \tau(gh_i)$ mean that $\tau(k) = \tau(e_0)$ for all $k \in G$. Thus $\tau$ is the trivial tensegrity in $T_{s,c}(\tilde{W};\Gamma)$ all of whose vertices coincide; i.e. $\tau = 1_G \otimes w$ for some $w \in \tilde{W}$. Since $\tau \in T^0_{s,c}(\tilde{W};\Gamma)$, it is in the kernel of the augmentation map and so $\tau$ must be the 0 tensegrity.

**Proposition 3.4** The evaluation map $e : \mathcal{R}[G] \otimes W \to W$ satisfies
\[
e \circ (L_k \otimes Id) = \rho(k) \circ e, \\
e \circ (R_k \otimes Id) = e \circ (Id \otimes \rho(k^{-1})).
\]

**Proof:**
\[
(e \circ (L_k \otimes Id))(\phi \otimes w) = \Sigma_{g \in G} (L_k \phi)(g)((\rho(g))w) \\
= \Sigma_{g \in G} \phi(k^{-1}g)((\rho(g))w) \\
= \Sigma_{g \in G} \phi(g)((\rho(kg))w) \\
= \Sigma_{g \in G} \phi(g)((\rho(k)\rho(k^{-1}))w) \\
= \rho(k)(e(\phi \otimes w))
\]

where the third line is obtained from the second by renaming the index of summation from $g$ to what was formerly $k^{-1}g$. Similarly
\[
(e \circ (R_k \otimes Id))(\phi \otimes w) = \Sigma_{g \in G} ((R_k \phi)(g))((\rho(g))w) \\
= \Sigma_{g \in G} \phi(kg)((\rho(g))w) \\
= \Sigma_{g \in G} \phi(g)((\rho(gk^{-1}))w) \\
= \Sigma_{g \in G} \phi(g)((\rho(g)\rho(k^{-1}))w) \\
= \rho(k)(e(\phi \otimes \rho(k^{-1})w))
\]

**Proposition 3.5** The inversion map $\text{Inv} \otimes Id : \mathcal{F}[G] \otimes W \to \mathcal{F}[G] \otimes W$ satisfies
\[
(\text{Inv} \otimes Id) \circ (R_k \otimes Id) = (L_k \otimes Id) \circ (\text{Inv} \otimes Id).
\]

**Proof:**
\[
(((\text{Inv} \otimes Id) \circ (R_k \otimes Id))(\phi \otimes w))(g) = (\text{Inv}(R_k \phi))(g)w
\]
\[(R_k \otimes Id) \circ (\text{Inv} \otimes Id)(\phi \otimes w)(g) = (R_k \otimes \text{Inv})(\phi \otimes w)(g) = \phi(g^{-1}k)w\]

Combining part 2) of Proposition 3.4 with the map \(i : W \to R[G] \otimes W\) gives another important identity:

**Proposition 3.6** If \(k \in G\), then right multiplication \(R_k\) satisfies

\[(R_k \otimes Id) \circ i = i \circ \rho(k^{-1}).\]

More generally, if \(\Lambda\) is an element of \(R[G]\) invariant by the inversion \(\text{Inv}\), then

\[(R_\Lambda \otimes Id) \circ i = i \circ \rho(\Lambda).\]

**Proof:**

\[((R_k \otimes Id) \circ i)w = (R_k \otimes Id)\sum_{g \in G} \delta_g \otimes \rho(g)w\]
\[= \sum_{g \in G} \delta_{gk} \otimes \rho(g)w\]
\[= \sum_{g \in G} \delta_{g} \otimes \rho(gk^{-1})w\]
\[= i(\rho(k^{-1})w)\]

For the second result, since \(\Lambda\) is symmetric in link set elements \(h_i\) and their inverses \(h_i^{-1}\),

\[((R_\Lambda \otimes Id) \circ i)w = (i \circ \rho(\text{Inv}\Lambda))w\]
\[= (i \circ \rho(\Lambda))w\]

We record below the result of left and right translation after Fourier Transform.

**Proposition 3.7** Suppose \(\rho\) is a representation of the finite group \(G\) on a vector space \(W\) and \(\phi \in R[G]\). Then the Fourier Transform \(\hat{\phi}\) satisfies

\[\hat{R_k}\phi(\rho) = \hat{\phi}(\rho) \circ \rho(k)\]
\[\hat{L_k}\phi(\rho) = \rho(k) \circ \hat{\phi}(\rho)\]
Proof:

\[
\hat{R}_k\phi(\rho) = \Sigma_{g \in G} \phi(gk^{-1}) \rho(g) \\
= \Sigma_{g' \in G} \phi(g') \rho(g'k) \\
= \hat{\phi}(\rho) \circ \rho(k)
\]

\[
\hat{L}_k\phi(\rho) = \Sigma_{g \in G} \phi(k^{-1}g) \rho(g) \\
= \Sigma_{g' \in G} \phi(g') \rho(kg') \\
= \rho(k) \circ \hat{\phi}(\rho)
\]

Below is a further equivariance property of the equivariant extension map \(i\).

**Proposition 3.8** Suppose \(\rho\) is a representation of the finite group \(G\) on a vector space \(W\) and \(i : W \rightarrow \mathcal{R}[G] \otimes W\) the associated equivariant extension map. Then

\[
(L_k \otimes \text{Id}) \circ i = (\text{Id} \otimes \rho(k^{-1})) \circ i
\]

Proof:

\[
((L_k \otimes \text{Id}) \circ i)(w) = (L_k \otimes \text{Id})\Sigma_{g \in G} (\delta_g \otimes \rho(g))(w) \\
= \Sigma_{g \in G} (\delta_{kg} \otimes \rho(g))(w) \\
= \Sigma_{g' \in G} (\delta_{g'} \otimes \rho(k^{-1}g'))(w) \\
= ((\text{Id} \otimes \rho(k^{-1})) \circ i)(w)
\]

The following is a nondegeneracy statement about the equivariant extension map.

**Proposition 3.9** Let \(e : \mathcal{R}[G] \otimes W \rightarrow W\) be the evaluation map, and \(i : W \rightarrow \mathcal{R}[G] \otimes W\) the equivariant extension map for a vector space \(W\) acted on by a representation \(\rho\) of \(G\). Then

\[
e \circ (\text{Inv} \otimes \text{Id}) \circ i(w) = |G|w
\]

where \(|G|\) is the order of the group \(G\).
Proof:

\[(e \circ (\text{Inv} \otimes \text{Id}) \circ i)(w) = (e \circ (\text{Inv} \otimes \text{Id})) \sum_{g \in G} \delta_g \otimes \rho(g)(w)
\]
\[= e \left( \sum_{g \in G} \delta_{g^{-1}} \otimes \rho(g)(w) \right)
\]
\[= \sum_{g \in G} \rho(g^{-1})\rho(g)(w)
\]
\[= |G|w.
\]

4 Bibliography


