Runge Kutta Behavior in the Presence of a Jump Discontinuity

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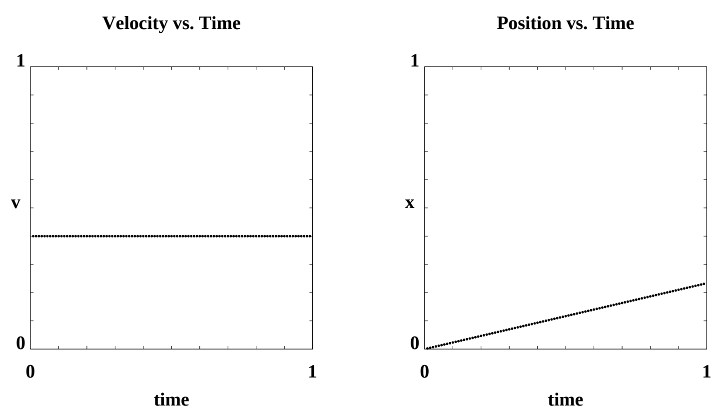
The graphs below appear to show motion at constant speed $v_0 = .4$ along a straight line. They were generated by numerical integration of a second order system of differential equations. But the slope of the straight line $x = x(t)$ is not the constant v_0! In fact the apparent velocity is $v_0 - \frac{1}{6}$.



The above system arose from numerical integration of

$$\begin{aligned} v' &= f(v) \\ x' &= v \end{aligned}$$

where $f(v) = -b\ sgn(v)$ and $b = 100$. By $sgn(v)$ we mean

$$sgn(v) = \begin{cases} 1 & \text{if } v > 0 \\ 0 & \text{if } v = 0 \\ -1 & \text{if } v < 0 \end{cases} .$$

The numerical integration method was fourth order Runge-Kutta with a step size of .01 and an initial value of $(v, x) = (.4, 0)$.

The Runge Kutta method (e.g. [**2**]) for the two dimensional system

$$(v', x') = F(t, v, x)$$

is derived under the assumption that F is three times differentiable. Here we see that the right hand side of the equation is not even a continuous function of v. So it is not surprising that Runge Kutta fails to give an accurate solution to this system. What is striking is the manner of its failure.

The following result shows that the behavior with the above initial condition always happens for initial values of v close enough to 0.

Theorem A: Let $b > 0$ and consider the system of ODE's

$$(v', x') = (-b\ sgn(v), v)$$

with initial condition $(v(t_0), x(t_0)) = (v_0, x_0)$. Then for any step size $h > 0$ with $0 < |v_0| < \frac{bh}{2}$, one step of fourth order Runge Kutta leaves v unchanged at v_0 and x changed from x_0 to

$$x_0 + h\left(v_0 - sgn(v_0)\ \frac{bh}{6}\right).$$

Since numerical integration preserves the condition $|v_0| < \frac{bh}{2}$, we can repeat as many steps as we like resulting in a numerical solution with constant velocity (v value) v_0 and constant *apparent velocity* (rate of change of x) of $(v_0 - \frac{bh}{6})$.

Intuitively, Runge Kutta involves making four preliminary estimates of where the next step ends up, each making use of what has been learned thusfar. We will refer to the process of obtaining the next preliminary estimate as a *substep* and use the superscripts ^{1st}, ^{2nd}, ^{3rd}, or ^{4th} to indicate which of the four preliminary estimates is being reported. We finalize the step by taking a suitable weighted average.

In referring to the details of the Runge-Kutta compution, we will also use the subscripts $_{IN}$, $_{MID}$, or $_{NXT}$. These indicate which time value (t_0, $t_0 + \frac{h}{2}$, or $t_0 + h$ respectively) a $\vec{y}$ value pertains to.

In vector form for the system $\vec{y}' = F(t, \vec{y})$ with initial condition $\vec{y}(t_0) = \vec{y}_{IN}$, the Runge-Kutta method may be expressed precisely as follows:

Substep	**Value of F**
1st	$\vec{F}^{1st} = F(t_0, \vec{y}_{IN})$
2nd	$\vec{F}^{2nd} = F(t_0 + \frac{h}{2}, \vec{y}^{1st}_{MID})$
3rd	$\vec{F}^{3rd} = F(t_0 + \frac{h}{2}, \vec{y}^{2nd}_{MID})$
4th	$\vec{F}^{4th} = F(t_0 + h, \vec{y}^{3rd}_{NXT})$

Substep	**Estimate of** $\vec{y}(t_0 + \frac{h}{2})$	**Estimate of** $\vec{y}(t_0 + h)$
1st	$\vec{y}^{1st}_{MID} = \vec{y}_{IN} + \frac{h}{2}\vec{F}^{1st}$	$\vec{y}^{1st}_{NXT} = \vec{y}_{IN} + h\vec{F}^{1st}$
2nd	$\vec{y}^{2nd}_{MID} = \vec{y}_{IN} + \frac{h}{2}\vec{F}^{2nd}$	$\vec{y}^{2nd}_{NXT} = \vec{y}_{IN} + h\vec{F}^{2nd}$
3rd		$\vec{y}^{3rd}_{NXT} = \vec{y}_{IN} + h\vec{F}^{3rd}$
4th		$\vec{y}^{4th}_{NXT} = \vec{y}_{IN} + h\vec{F}^{4th}$

The final resultant estimate of $\vec{y}(t_0 + h)$ given by one step in Runge-Kutta then is

$$\vec{y}_{NXT} = \frac{\vec{y}^{1st}_{NXT} + 2\vec{y}^{2nd}_{NXT} + 2\vec{y}^{3rd}_{NXT} + \vec{y}^{4th}_{NXT}}{6}.$$

With this description of the algorithm, we are ready for:

Proof of Theorem A: First note that replacing (v, x) by $(-v, -x)$ leaves the differential equation unchanged. Every step in Runge-Kutta also picks up a factor of -1, as does the conclusion of Theorem A. So once we have proven the theorem for $0 < v_0 < \frac{bh}{2}$, the case $0 > v_0 > -\frac{bh}{2}$ follows immediately.

We start by specializing the tabular description of one step in Runge-Kutta to the ODE system in Theorem A.

Substep	Value of F
1st	$\vec{F}^{1st} = (-b, v_0)$
2nd	$\vec{F}^{2nd} = (b, v_0 - \frac{bh}{2})$
3rd	$\vec{F}^{3rd} = (-b, v_0 + \frac{bh}{2})$
4th	$\vec{F}^{4th} = (b, v_0 - bh)$

Substep	Estimate of $\vec{y}(t_0 + \frac{h}{2})$	Estimate of $\vec{y}(t_0 + h)$
1st	$\vec{y}^{1st}_{MID} = (v_0 - \frac{bh}{2}, x_0 + \frac{hv_0}{2})$	$\vec{y}^{1st}_{NXT} = (v_0 - bh, x_0 + hv_0)$
2nd	$\vec{y}^{2nd}_{MID} = (v_0 + \frac{bh}{2}, x_0 + \frac{h}{2}(v_0 - \frac{bh}{2}))$	$\vec{y}^{2nd}_{NXT} = (v_0 + bh, x_0 + h(v_0 - \frac{bh}{2}))$
3rd		$\vec{y}^{3rd}_{NXT} = (v_0 - bh, x_0 + h(v_0 + \frac{bh}{2}))$
4th		$\vec{y}^{4th}_{NXT} = (v_0 + bh, x_0 + h(v_0 - bh))$

The proof of theorem A is now completed by inspecting the last column of the above table and observing that

$$\frac{1}{6}\left(\vec{y}^{1st}_{NXT} + 2\vec{y}^{2nd}_{NXT} + 2\vec{y}^{3rd}_{NXT} + \vec{y}^{4th}_{NXT}\right) = (v_0, x_0 + h(v_0 - \frac{bh}{6})).$$

What about other values of v_0? When $v_0 = 0$, numerical integration by this method will leave (v_0, x_0) unchanged; we have an equilibrium point of a discontinuous vector field. For $|v_0| > bh$, numerical integration avoids the line $v = 0$ and simply produces the expected answer $(v_0 - bh, x_0 + h(v_0 - \frac{bh}{2}))$. The other cases are covered below.

Theorem B: Let $b > 0$ and consider the system of ODE's

$$(v', x') = (-b\ sgn(v), v)$$

with initial condition $(v(t_0), x(t_0)) = (v_0, x_0)$. Then

a) For any step size $h > 0$ with $\frac{bh}{2} < |v_0| < bh$, one step of fourth order Runge Kutta changes (v_0, x_0) to

$$\left(v_0 - sgn(v_0)\ \frac{2bh}{3}, x_0 + h\left(v_0 - sgn(v_0)\ \frac{2bh}{3}\right)\right).$$

b) For a step size $h > 0$ with $v_0 = \pm bh$, one step of fourth order Runge Kutta changes (v_0, x_0) to

$$\left(\frac{v_0}{6}, x_0 + h\left(\frac{2v_0}{3}\right)\right).$$

c) For a step size $h > 0$ with $v_0 = \pm\frac{bh}{2}$, one step of fourth order Runge Kutta changes (v_0, x_0) to

$$\left(\frac{v_0}{3}, x_0 + h\left(\frac{v_0}{3}\right)\right).$$

The proofs are readily constructed by adjusting the entries in the proof of Theorem A to these slightly different v_0 values.

These answers are of a similar character to theorem A above. But since v_0 is not preserved, they only apply to one step and so will not be as striking if encountered. Also, since equality of real numbers is unstable in a floating point environment, there is no reason to expect the cases $v_0 = \pm bh$, $v_0 = \pm\frac{bh}{2}$, or $v_0 = 0$ to readily show up in practice.

Another question one may ask is what happens if the second order system is generalized to the d dimensional system in $\vec{y} = (y_1, y_2, \ldots, y_d)$:

$$\begin{aligned} y_1' &= -b\ sgn(y_1) \\ y_2' &= y_1 \\ \ldots &= \ldots \\ y_d' &= y_{d-1}. \end{aligned}$$

Here y_1 plays the role of the earlier v, and $y_2 = x$. And so y_{IN1} replaces what we earlier denoted by v_0.

Theorem D: Let $b > 0$ and y_{IN1} an initial velocity satisfying $0 < y_{IN1} < \frac{bh}{2}$. Then one step of fourth order Runge Kutta changes $\vec{y}$ from

$$\vec{y}_{IN} = (y_{IN1}, y_{IN2}, \ldots, y_{INd})$$

to

$$
\begin{aligned}
\vec{y}_{NXT} &= \vec{y}_{IN} \\
&+ h(0, y_{IN2}, y_{IN3}, \ldots, y_{INd-1}) \\
&+ h^2(0, -\frac{b}{6}, \frac{5}{6}y_{IN1}, \frac{5}{6}y_{IN2}, \ldots, \frac{5}{6}y_{INd-2}) \\
&+ h^3(0, 0, -\frac{b}{6}, \frac{1}{2}y_{IN1}, \frac{1}{2}y_{IN2}, \ldots, \frac{1}{2}y_{INd-3}) \\
&+ h^4(0, 0, 0, -\frac{b}{6}, \frac{1}{6}y_{IN1}, \frac{1}{6}y_{IN2}, \ldots, \frac{1}{6}y_{INd-4})
\end{aligned}
$$

With the notation we used earlier, the above is easy to establish once one expands the terms in the proof of Theorem A in powers of h. Recurrences are immediate which determine the h^p coefficients of Y_{NXTr} from the h^{p-1} coefficients of Y_{NXTr-1}.

The pictures in this note were made with the older software program dstool [**1**]. In fact the author first noted the phenomenon described here in the course of analyzing some "bug reports" (of a more complicated example) from that system.

References

[**1**] A. Back, J. Guckenheimer, M. Myers, F.J. Wicklin, P. Worfolk, *dstool: Computer Assisted Exploration of Dynamical Systems*, Notices of the American Mathematical Society 39 (1992) 303 – 309.
[**2**] J. Stoer, R. Bulirsch, *Introduction to Numerical Analysis*, Springer-Verlag, New York, 1980.