Some Parametrization Problems to Think About

Math 222

Curves
Think about a path (or possibly union of several paths) parametrizing each of the following curves.

Many of these are harder than we would expect you to come up with on an exam.

Lines in $\mathbb{R}^2$: The line between the points $(1, 1)$ and $(2, 4)$ in $\mathbb{R}^2$.

Boundary of a Polygon: The boundary of the trapezoid in $\mathbb{R}^2$ with consecutive vertices $(0, 0)$, $(1, 2)$, $(1, 3)$, and $(0, 2)$.

Lines in $\mathbb{R}^3$: The line between the points $(1, 1, 1)$ and $(2, 4, 8)$ in $\mathbb{R}^3$.

Graphs of functions: The parabola $y = x^2$ in $\mathbb{R}^2$ between the points $(-2, 4)$ and $(3, 9)$.

Circles in $\mathbb{R}^2$ trigonometrically: Centered at $(0, 0)$ and radius 2. Or centered at $(x_0, y_0)$ and radius $R$.

Circles in $\mathbb{R}^3$: Centered at $(0, 0, 0)$, radius 2, and lying in the plane $x + y + 2z = 4$.

Hint: Suppose you knew two orthogonal unit vectors perpendicular to the normal to the plane?

A Standard Ellipse trigonometrically: $4x^2 + 9y^2 = 16$.

A Standard Ellipse as union of graphs: $4x^2 + 9y^2 = 16$.

A Rotated Ellipse: $4(x + y)^2 + 9(x - y)^2 = 16$ or the off center version, $4(x + y + 1)^2 + 9(x - y + 3)^2 = 16$.

Curve of intersection of surfaces: The curve of intersection of the cylinder $x^2 + y^2 = 1$ and the paraboloid $z = x^2 + 4y^2$.

A Standard Hyperbola: $x^2 - y^2 = 1$ as a union of graphs.
A Standard Hyperbola: \( x^2 - y^2 = 1 \) via hyperbolic functions. Remark:
The functions \( \cosh \) and \( \sinh \) defined by
\[
\cosh x = \frac{e^x + e^{-x}}{2} \\
\sinh x = \frac{e^x - e^{-x}}{2}
\]
satisfy \( \cosh^2 x - \sinh^2 x = 1 \) and have many properties analogous to the
trigonometric functions \( \cos x \) and \( \sin x \).

Another Hyperbola: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \).

Geometric Descriptions: The path traced out by a point on a circular
wheel rolling without slipping along a straight line.

A Curve in polar coordinates: The four leaved rose given in polar coor-
dinates as \( r = \cos(2\theta) \).

A Curve in spherical coordinates: The meridian \( \theta = \frac{\pi}{4} \) lying on the unit
sphere of radius 1 centered at the origin.

Surfaces
Think about a parametrization \( \Phi : U \subset R^2 \to R^3 \) (or possibly a union of such)
for each of the following surfaces: Some are quite hard.

A Plane: \( x + y + 2z = 2 \). Or the portion of this plane lying in the first
octant.

The Graph of a function: The portion of the paraboloid \( z = x^2 + 4y^2 \)
where \( z \leq 4 \). Or \( x = z^2 + 4y^2 \).

A Paraboloid trigonometrically: \( z = x^2 + y^2 \). Or \( z = x^2 + 4y^2 \).

A Triangle as union of graphs: The triangle (including its interior) with
vertices \((1, 0, 0), (1, 2, 0), \) and \((2, 0, 3)\).

Surfaces of Revolution: The paraboloid in \( R^3 \) obtained by rotating the
parabola \( y = x^2 \) in \( R^2 \) about the \( y \) axis.
Or the surface obtained by rotating this parabola about the \( x \) axis.
Or the surface obtained by rotating about the \( y \) axis the curve in \( R^2 \)
described by \( c(t) = (\sin t, e^t) \) for \( 0 \leq t \leq \pi \).
Note the techniques here generalize immediately to any graph or path in
\( R^2 \).

Sphere Using spherical coordinate ideas: \( x^2 + y^2 + z^2 = 9 \). Or \( x^2 +
2x + y^2 + 4y + z^2 + 6z = 0 \). Or the ellipsoid \( x^2 + 4y^2 + 9z^2 = 36 \).
Sphere Using cylindrical coordinate ideas: Same examples.

A Sphere (or Ellipsoid) as union of several graphs: Same examples.

Right Circular Cylinder trigonometrically: \( x^2 + z^2 = 4 \).

Right Circular Cylinder as union of several graphs: \( x^2 + y^2 = 4 \). Or the portion above the \( xy \) plane and below the plane \( z = x + y + 4 \).

Hyperboloid: \( x^2 + y^2 - z^2 = 1 \) as union of several graphs or using cylindrical coordinate ideas.

Hyperboloid using hyperbolic (and trigonometric) functions: Same example, but using the idea behind the hyperbolic function parametrization of the hyperbola \( x^2 - y^2 = 1 \) in \( \mathbb{R}^2 \).

A Right Circular Cone: \( z^2 = x^2 + y^2 \) with \( z \geq 0 \).

Hyperbolic Cylinder in \( \mathbb{R}^3 \): \( x^2 - y^2 = 1 \).

Extruded Curve: This is a more general form of cylinder. Let \( c(t) = (x(t), y(t), 0) \) for \( a \leq t \leq b \) be a path in the \( xy \) plane. (e.g. \( c(t) = (\cos t, \sin t, 0) \) would describe a unit circle.) Consider the surface obtained by moving each point of this curve a distance 4 perpendicular to the \( xy \) plane in the positive \( z \) axis direction. (This would give a right circular cylinder for the specific \( c(t) \) above.) Or move each point a signed distance \( f(t) \) along this line. One can also start with a curve in an arbitrary plane of \( \mathbb{R}^3 \) and move along a line not perpendicular to this plane.

A Triangle via barycentric coordinates: The triangle (including its interior) with vertices \( P_1 = (1, 0, 0) \), \( P_2 = (1, 2, 0) \), and \( P_3 = (2, 0, 3) \).

HINT: Can you justify why \( \{t_1 P_1 + t_2 P_2 + t_3 P_3 : \text{all } t_i \geq 0 \text{ and } t_1 + t_2 + t_3 = 1\} \) describes the points of the triangle with vertices \( P_1 \), \( P_2 \), and \( P_3 \)? Do you see analogues for line segments and tetrahedra?

A Triangle as an intersection: The intersection with the plane \( z = 2 \) of the (solid) tetrahedron whose vertices are \( (0, 0, 0) \), \( (1, 0, 0) \), \( (1, 2, 0) \), and \( (0, 0, 5) \).

Hint: This plane is parallel to one of the faces of the tetrahedron.