Why is the Variance of the Sum of Two Independent Random Variables the Sum of the Variances?

Imagine two such random variables X and Y.

X	**probability**
x_1	p_1
x_2	p_2
$\cdots$	$\cdots$
x_n	p_n

Y	**probability**
y_1	q_1
y_2	q_2
$\cdots$	$\cdots$
y_m	q_m

Since X and Y are independent random variables, the probability of X taking on the value x_i and Y the value y_j is simply the product $p_i q_j$.

Below is the table describing the random variable $X + Y$. Some values may appear more than once in the nm rows below, but this does not throw off our formulae for the variance.

$\mathbf{X + Y}$	**probability**
$x_1 + y_1$	$p_1 q_1$
$x_1 + y_2$	$p_1 q_2$
$\cdots$	$\cdots$
$x_n + y_m$	$p_n q_m$

Now

$$\begin{aligned}
Var(X+Y) &= \Sigma_{i,j} p_i q_j (x_i + y_j - \mu_X - \mu_Y)^2 \\
&= \Sigma_{i,j} p_i q_j (x_i - \mu_X)^2 + \Sigma_{i,j} p_i q_j (y_j - \mu_Y)^2 + 2\Sigma_{i,j} p_i q_j (x_i - \mu_X)(y_j - \mu_Y) \\
&= (\Sigma_j q_j)\, Var(X) + (\Sigma_i p_i)\, Var(Y) + 2\,(\Sigma_i p_i (x_i - \mu_X))\,(\Sigma_j q_j (y_j - \mu_Y)) \\
&= 1 \cdot Var(X) + 1 \cdot Var(Y) + 2 \cdot 0 \cdot 0 \\
&= Var(X) + Var(Y).
\end{aligned}$$