INTEGRATION BY PARTS V3

Much of this is making explicit how divergence computations (easy in index notation) look in more general notations. The index point of view is built on the fact that the integral of the divergence of a vector field over a manifold without boundary is zero.

Matrix Basis Elements and Brackets.

1: \( E^j_i \) has entry 1 in row \( i \), column \( j \) - 0 elsewhere; \( E^j_i e_k = \delta^j_k e_j \) where \( O^j_i = E^j_i - E^i_j \).

2: \[ E^j_i, E^k_l \] = \(-\delta^l_i E^j_k + \delta^j_k E^l_i \).

3: \[ O^j_a, O^l_e \] = \(-\delta_{ae} O^j_b + \delta_{ad} O^c_b + \delta_{be} O^d_a - \delta_{bd} O^c_a \).

4: \[ E^j_i, E^k_l \] = \( c^i_{jk} E^i \) \( \iff d\theta^i = -\frac{1}{2} c^i_{jk} \theta^j \wedge \theta^k \).

Differential Form Conventions.

1: \( \theta_1 \wedge \theta_2 = \frac{1}{2} (\theta_1 \otimes \theta_2 - \theta_2 \otimes \theta_1) \)

2: \( 2d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \)

3: \begin{align*}
\sum_{ij} \alpha_{ij} \theta_i \otimes \theta_j &= \alpha_{12} \theta_1 \wedge \theta_2 = \frac{1}{2} \begin{pmatrix} 0 & \alpha_{12} \\ -\alpha_{12} & 0 \end{pmatrix} \\
\theta^1 \wedge \theta^2 \ldots \theta^k, \bar{\theta}^1 \wedge \bar{\theta}^2 \ldots \bar{\theta}^k > &= \det(\theta^i \cdot \theta^j) \\
\text{(So k! times the natural tensor inner product.)}
\end{align*}

4: \( \ast \ast = (-1)^{p(n-p)} \).

5: \begin{align*}
\langle \alpha, \beta \rangle d\mu &= \alpha \wedge \ast \beta \\
\end{align*}

6: \begin{align*}
\delta &= (-1)^{np+n+1} \ast \text{d} \ast \text{on p-forms since} \\
d(\alpha^{p-1} \wedge \ast \beta^p) &= d\alpha^{p-1} \wedge \ast \beta^p + (-1)^{p-1} \alpha^{p-1} \wedge d \ast \beta^p \\
 &= d\alpha^{p-1} \wedge \ast \beta^p + (-1)^{p-1} \alpha^{p-1} \wedge (-1)^{(p-1)(n-p+1)} \ast d \ast \beta^p \\
\text{and } (p-1)(n-p) &= np + n \mod 2.
\end{align*}
7: Notation and *
\[ \omega = \begin{pmatrix} \omega_1 & \omega_2 \end{pmatrix} = \Sigma \omega_i \theta^i \]
\[ \ast \omega = \begin{pmatrix} -\omega_2 & \omega_1 \end{pmatrix} \]
\[ \omega = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \end{pmatrix} = \Sigma \omega_i \theta^i. \]
\[ \ast \omega = \frac{1}{2} \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \]

\(R^2\) or \(R^3\) Case.
1: Notation for vectors (type(1,0)):
\[ \vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \Sigma v^i e_i. \]
2a: Notation for matrices (type(1,1)):
\[ M = \begin{pmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{pmatrix} = (\vec{M}_1 \quad \vec{M}_2) = \Sigma M^i_j E^j_i. \]
(Upper index of \(M\) is the row index.)
2b: Notation for bilinear form (type(0,2)) case:
\[ Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \Sigma Q_{ij} \theta^i \otimes \theta^j. \]
3: \(< \vec{v}, \vec{w} >= v^i w^i = v_i w^i. \)
4: \(< A, B >= tr(A^t B) = A^i_j B^j_i. \)
5: \(\text{div}(\vec{v}) = \partial_i v^i = \partial_1 v^1 + \partial_2 v^2. \)
6: Divergence on \(M\) of type (1,1) giving result of type (1,0):
\[ \text{div}(M) = (\text{div}(\vec{M}_1) \quad \text{div}(\vec{M}_1)) = \Sigma (\partial_j M^i_j) e_i. \]
7: \((\partial_i f)g = \text{div}(f \vec{g} e_i) - f \partial_i g. \)
8a: \((\partial_i f) v^i = \partial_i (f v^i) - f \partial_i v^i. \)
8b: \(< \nabla f, \vec{v} >= \text{div}(f \vec{v}) - f \text{div}(\vec{v}). \)
9a: \(M_j^i (\partial_i v^j) = \partial_i (M_j^i v^j) - v^j \partial_i M_j^i. \)
9b: \(tr(M_j^i D \vec{v}) = < D \vec{v}, M^i_j >= \text{div}(M \vec{v}) - (\text{div}M) \vec{v}. \)

Interior Product and Lie Derivative Conventions.
1: \((\theta_1 \wedge \ldots \wedge \theta_p)(e_1, \ldots, e_p) = \frac{1}{p!} e^i_1 \ldots e^i_p. \)
2: \(i_X(\alpha^p)(Y_2, \ldots, Y_p) = p \alpha(X, Y_2, \ldots, Y_p). \)
3: \(i_{e_i}(\theta_1 \wedge \ldots \wedge \theta_p) = (\theta_2 \wedge \ldots \wedge \theta_p). \)
4: \(i_X(\alpha^p \wedge \beta^q) = i_X(\alpha^p) \wedge \beta^q + (-1)^p \alpha^p \wedge i_X(\beta^q). \) (An anti-derivation, like \(d\).)
5: \(L_X = i_X \circ d + d \circ i_X \) by checking on \(\alpha^p \wedge \beta^q\) and using induction on the total degree.
6: \((L_X \alpha^p)(Y_1, \ldots, Y_p) = (\nabla_X \alpha^p)(Y_1, \ldots, Y_p) + \sum_i \alpha^p(Y_i, \ldots, \nabla Y_i, \ldots, Y_p)\).
This is because \(\nabla X Y_i - L_X Y_i = \nabla Y_i X\).

7: The formulas
\[L_X Y = \nabla_X Y - \nabla_Y X\]
\[L_X(\theta)(Y) = (\nabla_X \theta)(Y) + \theta(\nabla_Y X)\]
and their extensions to tensors of type \(T^s_r\) correspond to
\[(L_X T)_{\alpha}^{ij} = X^c \nabla_c T_{\alpha}^{ij} - \sum_{k=1}^s T_{\alpha}^{ij12...ci} \nabla_c X^k\]
\[+ \sum_{k=1}^r T_{\alpha}^{ij12...cj} \nabla_j X^k\]

**Divergences in Special Cases.**

1: With the definition \(\text{div}(X) d\mu = L_X d\mu\), we also have
\[\text{div}(X) = \nabla_i X^i = \theta^i(\nabla e_i X) = \text{trace}(Y \rightarrow \nabla Y X)\]
since for an orthonormal coframe field and each \(i\)
\[(L_X \theta^i) e_i = (\nabla_X \theta^i) e_i + \theta^i(\nabla e_i X)\]
and \((\nabla_X \theta^i) e_i = -\theta^i(\nabla e_i X) = - \langle e_i, \nabla_X e_i \rangle = 0\).

2: Above \(\text{div}\) is the negative of the formal adjoint to grad because
\[\text{div}(X) = \text{trace}(Y \rightarrow \nabla Y X) = \sum \langle \nabla e_i X, e_i \rangle\]
implies
\[\text{div}(f X) = \langle \nabla f, X \rangle + f \text{div}(X)\,.
\]
(This used \(\sum (e_i(f)) \langle X, e_i \rangle = X f\).)

3: For a 1-form \(\omega\), the definition
\[\delta \omega = -\text{trace}(Y \rightarrow \nabla Y \omega) = -\sum(\nabla e_i \omega)(e_i)\]
leads to
\[\delta(f \omega) = f \delta \omega - \langle df, \omega \rangle\]
since \(\langle df, \omega \rangle = \sum(e_i(f)) \omega(e_i) = X f\), \(X\) being the vector field dual to \(\omega\). Thus all the following are equal in this case:
\[X f = \langle df, \omega \rangle = \langle \nabla f, X \rangle = \omega(\nabla f)\,.
\]
Also
\[\delta \omega = -\text{div} X\]
since starting with \(\omega = \langle X, \cdot \rangle\), we have
\[\nabla e_i \omega = \langle \nabla e_i X, \cdot \rangle\]
\[\sum(\nabla e_i \omega)(e_i) = \sum \langle \nabla e_i X, e_i \rangle\,.
\]

4: With K-N conventions of
\[\nabla e_i e_j = \Gamma^k_{ij} e_k\]
\[\nabla e_i \theta^k = -\Gamma^k_{ij} \theta^j\]
\[\Gamma^k_{ij} = -\Gamma^k_{ik}\]
for an orthonormal basis, we have
\[
\nabla e_i \Sigma_j e_j \otimes e_j = \Sigma_{j,k} \Gamma^k_{ij} e_k \otimes e_j + \Gamma^k_{ij} e_j \otimes e_k
\]
\[
= \Sigma_{j,k} \Gamma^k_{ij} e_k \otimes e_j + \Gamma^j_{ik} e_k \otimes e_j
\]
\[
= 0
\]
since \( \Gamma^k_{ij} = -\Gamma^j_{ik} \).

5: So starting with \( X = \Sigma_j (\omega(e_j)) e_j \), we have
\[
\nabla e_i X = \Sigma_j \nabla e_i (\omega(e_j)) e_j
\]
\[
div X = \Sigma_i < \nabla e_i X, e_i > = \Sigma_i < ((\nabla e_i \omega)(e_j)) e_j, e_i >
\]
\[
= \Sigma_i (\nabla e_i \omega)(e_i) + 0
\]
\[
= -\delta \omega
\]

where \( C^1 \) is contraction on the first two slots and the second term vanishes because \( \nabla e_i \Sigma_j e_j \otimes e_j = 0 \).

6: As in Besse (but with the K-N \( T^s_r \) convention which is opposite to Besse), view
\[
\nabla : T^r_s \rightarrow \Omega^1(M) \otimes T^r_s
\]
so that the “slot” for the covariant derivative comes first. Then
\[
\nabla (A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B).
\]

7: Use the notation
\[
C_{i_1,j_1} \cdots C_{i_r,j_r} T
\]
for contraction pairing covariant indices \( i_1 \) with \( j_1 \), etc. Similarly for contravariant or mixed contractions. Note with this notation
\[
C_{1,2}C_{3,4} T = C_{1,2} (C_{1,2} T) = C_{1,2} (C_{3,4} T)
\]
The above would also be \( C_{1,2} (C_{3,4} T) \). This notation lines up with natural contractions to write in index notation.

8: We use the non-standard notation \( I_{i,j} \) to indicate the interchange of covariant slots \( i \) and \( j \) in a tensor. With this notation, for a 1-form \( \omega \), the symmetrized covariant derivative is
\[
\delta^* \omega = \frac{1}{2} (\nabla \omega + I_{1,2} (\nabla \omega)).
\]

9: The symmetrized covariant derivative from 1-forms to symmetric covariant 2-tensors (sections of) \( S^2(T^* M) \) is defined by
\[
(\delta^* \omega)(X,Y) = \frac{1}{2} ((\nabla_X (\omega)) + (\nabla_Y (\omega)))
is the dual to the divergence $\delta : S^2(T^*M) \to T^*(M)$ defined by

$$\delta h = -\text{trace}((Y, Z) \to (\nabla h)(Y, Z, \ )$$

$$\delta h (X) = -\text{trace}((Y, Z) \to (\nabla h)(Y, Z, X)$$

$$\delta h (X) = -\Sigma_i(\nabla e_i h)(e_i, X).$$

To see this, note

$$-\delta (C_{1,2} (\omega \otimes h)) = C_{1,4}C_{2,3}((\nabla \omega) \otimes h)$$

$$+C_{1,4}C_{2,3}(\omega \otimes (\nabla h))$$

$$= \frac{1}{2} (C_{1,4}C_{2,3} + C_{1,3}C_{2,4}) ((\nabla \omega) \otimes h)$$

$$- < \omega, \delta h >$$

$$= \frac{1}{2} (C_{2,4}C_{1,3}) ((I_{1,2} (\nabla \omega)) \otimes h)$$

$$+ \frac{1}{2} (C_{2,4}C_{1,3}) ((\nabla \omega) \otimes h)$$

$$- < \omega, \delta h >$$

$$= - < \delta^* \omega, h >$$

$$= - < \omega, \delta h >$$

where we have used the argument interchange operator $I_{1,2}$ as well as the symmetry of $h$.

10: For a 1-form $\omega$,

$$\delta^* \omega = \frac{1}{2} L_X g$$

where $X$ is the 1-form dual to $\omega$ and $g = < , >$ is the inner product. This is because

$$(L_X(g))(U, V) = (\nabla_X g)(U, V) + < \nabla_U X, V > + < U, \nabla_V X >$$

$$= 2 (\nabla \omega + I_{1,2} (\nabla \omega)) (U, V)$$

General Case.

1: Besse Formulation is $\nabla : T^*_r \to \Omega^1(M) \otimes T^*_r$ has formal adjoint $\nabla^* : \Omega^1(M) \otimes T^*_r \to T^*_r$ given by

$$(\nabla^* \alpha)(X_1, \ldots, X_r) = -\Sigma(\nabla Y_i \alpha)(Y_i, X_1, \ldots, X_r)$$

for an orthonormal basis $Y_i$. The “opposite” (as in minus sign) of the trace of

$$(X, Y) \to (\nabla X \alpha)(Y, X_1, \ldots, X_r).$$

The trace here might also be referred to as the contraction $C_{1,2}$.
2: To prove this, for $\alpha \in T_{r+1}^s$ and $\beta \in T_r^s$,
\[-\delta (C_1 (\beta \otimes \alpha)) = C_{1,1} (\nabla (C_1 (\beta \otimes \alpha)))
= C_2 ((\nabla \beta \otimes \alpha) + (\beta \otimes \nabla \alpha))
= < \nabla \beta, \alpha > + < \beta, C_{1,2} \nabla \alpha >
= < \nabla \beta, \alpha > - < \beta, \nabla^* \alpha >
\]
where the $C_i$ are contractions. Specifically
\[
C_1 = C_{1,r+2} \ldots C_{r,2r+1} C_{1,s+1} \ldots C_{s+1,2s}
C_2 = C_{1,r+2} \ldots C_{r+1,2r+2} C_{1,s+1} \ldots C_{s+1,2s}
\]
So $C_{1,2} (C_1 T) = C_2 T$ for a tensor field $T \in T_{r+1}^s$.

3: $\nabla^*$ is also the divergence $\delta$ on forms since
\[
< d\alpha, \beta > = < \text{Alt}(\nabla \alpha), \beta >
= < \nabla \alpha, \text{Alt}(\beta) >
= < \nabla \alpha, \beta >
= < \alpha, \nabla^* \beta >
\]
using the facts that $\beta$ and $\nabla^* \beta$ are skew symmetric, and the fact that the inner product on forms is the same (or a multiple of) the inner product on tensors.