Differential Geometry Conventions

1. E_i^j has entry 1 in row i, column j - 0 elsewhere; $E_i^j e_k = \delta_k^i e_j$ where $O_i^j = E_i^j - E_j^i$.
2. $[E_i^j, E_l^k] = -\delta_i^l E_k^j + \delta_k^j E_i^l$.
3. $[O_a^b, O_c^d] = -\delta_{ac} O_b^d + \delta_{ad} O_b^c + \delta_{bc} O_a^d - \delta_{bd} O_a^c$.
4. $[E_j, E_k] = c_{jk}^i E_i \Longleftrightarrow d\theta^i = -\frac{1}{2} c_{jk}^i \theta^j \wedge \theta^k$.
5. $d\omega(X, Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X, Y)$.
6. $d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i$.
7. $d\theta(X, Y) = -\frac{1}{2}(\omega(X)\theta(Y) - \omega(Y)\theta(X)) + \Theta(X, Y)$.
8. $d\theta^i = -\omega_j^i \wedge \theta^j + \Theta^i$.
9. $dw = -\frac{1}{2}[\omega, \omega] + \Omega$.
10. $D\Omega = 0$, $d\Omega = [\Omega, \omega]$.
11. $d\Omega_j^i = \Omega_k^i \wedge \omega_j^k - \omega_k^i \wedge \Omega_j^k$.
12. $D\Theta = \Omega \wedge \theta$.
13. $T(X, Y) = u(2\Theta(X^*, Y^*))$, $R(X, Y)Z = u(2\Omega(X^*, Y^*)u^{-1}Z)$.
14. C { $R(X, Y)Z\} = \mathcal{C}\{T(T(X, Y), Z) + \nabla_X T(Y, Z)\}$.
15. $\mathcal{C}\{(\nabla_X R)(Y, Z) + R(T(X, Y), Z)\} = 0$.
16. $\Omega_j^i = \frac{1}{2} R_{jkl}^i \theta^k \wedge \theta^l$.
17. Local trivialization $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$; Transition function $\psi_{\beta\alpha} = \phi_\beta \phi_\alpha^{-1}$
18. $\omega_\beta = Ad(\psi_{\alpha\beta}^{-1})\omega_\alpha + \theta_{\alpha\beta}$ where $\theta_{\alpha\beta} = \psi_{\alpha\beta}^* \theta$.
19. $\sigma^*\omega = \Gamma_{jk}^i dx^j E_i^k$.

20. Left invariant Maurer-Cartan form θ is $s^{-1}ds$ on $Gl(n)$.

21. $\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^i_{jk} \frac{\partial x^j}{\partial \bar{x}^\beta} \frac{\partial x^k}{\partial \bar{x}^\gamma} \frac{\partial \bar{x}^\alpha}{\partial x^i} + \frac{\partial^2 x^i}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} \frac{\partial x^\alpha}{\partial x^i}$.

22. $\partial_k H^i_j = D_k H^i_j + H^l_j \Gamma^i_{kl} - H^i_l \Gamma^l_{kj}$.

23. $R(X_k, X_l)X_j = R^i_{jkl} X_i$.

24. $R^i_{jkl} = D_k \Gamma^i_{lj} - D_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm}$.

25. $2\langle \nabla_Y Z, X\rangle = Y\langle X, Z\rangle + Z\langle Y, X\rangle - X\langle Y, Z\rangle - \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle + \langle X, [Y, Z]\rangle$.

26. $g_{lk}\Gamma^l_{ji} = \frac{1}{2}(D_j g_{ki} + D_i g_{jk} - D_k g_{ji})$.

27. $R(v_1, v_2, v_3, v_4) = \langle R(v_3, v_4)v_2, v_1\rangle$.

28. $\kappa = \frac{R(v_1,v_2,v_1,v_2)}{|v_1 \wedge v_2|^2}$; $S(v_1, v_2) = \Sigma^n_{i=1} R(e_i, v_1, e_i, v_2)$.

29. $R(v_1, v_2, v_3, v_4) = R(v_3, v_4, v_1, v_2) = -R(v_2, v_1, v_3, v_4)$.

30. $s_i = X^j_i \frac{\partial}{\partial x^i} \Leftrightarrow \theta^i = Y^i_j dx^j$ where $Y = X^{-1}$.

31. $\omega^i_j = s^{-1}ds + Ad(s^{-1})\omega_e = Y^i_j dX^k_j + \Gamma^k_{ml} X^l_j dx^m$.

32. $\langle \theta^1 \wedge \theta^2 \ldots \theta^k, \bar{\theta^1} \wedge \bar{\theta^2} \ldots \bar{\theta^k}\rangle = \det(\theta^i \cdot \theta^j)$.

33. $\langle \psi \wedge \theta, \bar{\psi} \wedge \bar{\theta}\rangle = \frac{r!s!}{(r+s)!}\langle \psi, \bar{\psi}\rangle\langle \theta, \bar{\theta}\rangle$ if $\psi, \bar{\psi} \in A^r(V_1)$ and $\theta, \bar{\theta} \in A^r(V_2)$ and $V_1 \perp V_2$.

34. $\alpha \wedge *\beta = k!\langle \alpha, \beta\rangle\omega$ where $\omega = dvol$ and $\alpha, \beta \in A^k$.

35. $i_Y(\alpha) = kC(Y \otimes \alpha) = k\alpha(Y, \cdots)$ where $\alpha \in A^k$. The map i_Y is an anti-derivation.

36. $** = (-1)^{k(n-k)}$.

37. $\delta = (-1)^{k(n-k)+1} * d*$; $\langle \delta\alpha, \beta\rangle = k\langle \alpha, d\beta\rangle$ where $\delta : A^k \to A^{k-1}$.

38. $L^*_X = (-1)^{k(n-k)+1} * L_X *$.

39. $(k + 1)\langle \psi \wedge \alpha, \beta\rangle = \langle \alpha, i_X \beta\rangle$ where ψ is dual to X and $\alpha \in A^k$.

40. $*(\psi \wedge *\alpha) = (-1)^{nk} i_X \alpha$ where $\alpha = \theta^i \wedge i_{X_i}\alpha$.

41. $\delta(f\alpha) = f\delta\alpha - i_{\nabla f}\alpha$.

42. $d\alpha = (-1)^k Alt(\nabla\alpha)$; $\delta\alpha - kC(1, k + 1)\nabla\alpha$.

43. $L^*_X = \delta \circ \psi \wedge +\psi \wedge \circ\delta$.

44. $*L_X\omega = -\delta\psi$.

45. $L_X^* = -(L_X + h_X - divX)$ where $k!\langle h_X\alpha,\beta\rangle = L_X\langle,\rangle(\alpha,\beta)$. Alternatively $\beta \wedge *h_X\alpha = L_X\langle,\rangle(\alpha,\beta)$.

46.

$$\begin{aligned} \nabla_X Y &= \nabla_X Y + \alpha(X,Y)\ (Gauss) \\ \nabla'_X\xi &= -A_\xi X + D_X\xi\ (Weingarten) \\ \langle A_\xi X, Y\rangle &= \langle \xi, \alpha(X,Y)\rangle \end{aligned}$$

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47. $R'(W,Z,X,Y) = R(W,Z,X,Y) + \langle\alpha(X,Z),\alpha(Y,W)\rangle - \langle\alpha(Y,Z),\alpha(X,W)\rangle$ (Gauss).

48. $(R'(X,Y)Z)_\perp = (\tilde{\nabla}_X\alpha)(Y,Z) - (\tilde{\nabla}_Y\alpha)(X,Z)$ (Codazzi).

49. $(\tilde{\nabla}_X\alpha)(Y,Z) = D_X(\alpha(Y,Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$.

50. $X'' + \nabla_T(\mathcal{T}(X,T)) + R(X,T)T = 0$.

51. X a JVF, $\langle X', Y\rangle \mid_a^b = \int_a^b \langle X', Y'\rangle - \langle R(X,T)T, Y\rangle dt = I_a^b(X,Y)$.

52. $\frac{d}{ds}L(\tau^s) \mid_{s=0} = \langle X,T\rangle \mid_a^b - \int_a^b \langle X, \nabla_T T\rangle$ where τ^s is parameterized proportional to arc length and τ is parameterized by arc length.

53. $X,Y \perp \tau$, piecewise smooth, and 0 at endpoints $\Rightarrow \frac{d^2}{ds^2}L(\tau^s) \mid_{s=0} = I_a^b(X,Y)$.

54. $A_X = L_X - \nabla_X$, $u_0 \circ \Lambda_X \circ u_0^{-1} = (-A_X)_0$.

55. $T(X,Y)_0 = u_0 \circ \Lambda(X) \circ u_0^{-1} Y_0 - u_0 \circ \Lambda(Y) \circ u_0^{-1} X_0 - [X,Y]_0$.

56. $R(X,Y)_0 = u_0 \circ ([\Lambda_X, \Lambda_Y] - \Lambda([X,Y])) \circ u_0^{-1}$.

57. $\Lambda_{\mathcal{M}}(X)Y = \frac{1}{2}[X,Y]_{\mathcal{M}} + U(X,Y)$.

58. $2B(U(X,Y),Z) = B(X,[Z,Y]_{\mathcal{M}}) + B([Z,X]_{\mathcal{M}},Y)$.

59. $(L_X * - * L_X)\alpha = (*h_X - divX*)\alpha$.

60. θ^i orthonormal $\Rightarrow \delta\theta^i = -\Gamma^j_{ji}$.

61. $\delta(\theta^i \wedge \theta^j) = (\Gamma^k_{ji} - \Gamma^k_{ij})\theta^k - \Gamma^k_{ki}\theta^j + \Gamma^k_{kj}\theta^i$.

62. $\delta(\theta^i \wedge \theta^j \wedge \theta^k) = -\Gamma^p_{pk}\theta^i \wedge \theta^j + \Gamma^p_{pj}\theta^i \wedge \theta^k - \Gamma^p_{pi}\theta^j \wedge \theta^k + (\Gamma^p_{ki} - \Gamma^p_{ik})\theta^j \wedge \theta^p + (\Gamma^p_{jk} - \Gamma^p_{kj})\theta^i \wedge \theta^p + (\Gamma^p_{ij} - \Gamma^p_{ji})\theta^k \wedge \theta^p$.

63. $D_p\Gamma^u_{qi} = \frac{1}{3}(R_{qpiu} + R_{ipqu})$ at 0 in normal coordinates.