CARTAN FOR BEGINNERS CHAPTER 4

V7 AB

0.1. Tableaux.

1: Seeking maps $f : V^n \to W^s$. $1 \leq i, j, k \leq n$, $1 \leq a, b, c \leq s$.
2: V^n has basis $\{v_i\}$, W^s has basis $\{w_a\}$. $x = x^i v_i \in V$. $u = u^a w_a \in W$.
3: Dual bases $\{v^i\}$,$\{w^a\}$.
4: Given map $f : V \to W$, Gauss map:

$$\gamma_f : V \to W \otimes V^* = \mathrm{Hom}(V, W)$$
$$\gamma_f = \frac{\partial f^a}{\partial x^i} w_a \otimes v^i.$$

5: Or $\gamma_f = \partial_i u^a w_a \otimes v^i$.
6: $W \otimes V^* = J^1_x(V, W)/J^0_x(V, W)$.
7: R homog. cst coeff eqns for $1 \leq r \leq R$:

$$B^{ri}_a \frac{\partial f}{\partial x^i} = 0.$$

8: Equivalently $< b, \gamma_f >= 0$ $\forall b \in B \subset W^* \otimes V$. *(Contraction.)*
B is the space of *symbol relations.*
9: C-R Eqns Example: $\partial_1 u^1 - \partial_2 u^2 = 0$, $\partial_1 u^2 + \partial_2 u^1 = 0$ corresponds to $B = \{w^1 \otimes v_1 - w^2 \otimes v_2, w^2 \otimes v_1 + w^1 \otimes v_2\}$
10: $B^\perp \subset W \otimes V^*$ is the space of **admissable first derivatives**.
11: A tableau A is a linear subspace $A \subset W \otimes V^*$.
12: In the C-R case

$$B^\perp = \{w_1 \otimes v^1 + w_2 \otimes v^2, -w_2 \otimes v^1 + w_1 \otimes v^2\}$$

or as matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(This agrees with $w_s \otimes v^i$ representing the matrix with 1 in row i column s.)
13: $A = 0$ implies solution depends on s constants.
14: $A = W \otimes V^*$ implies solution depends on s functions of n variables.
15: $A = W \otimes L^*$ with e.g. $L^* = \{v^1, \ldots, v^k\}$ implies solutions depend on s functions of k variables.

$$A = \begin{pmatrix} * & 0 \end{pmatrix}$$

in block form.

16: $A = Y \otimes V^*$ with Y e.g. $\{w_1, \dots, w_p\}$implies solutions depend on p functions of n variables and $s - p$ constants.

$$A = \begin{pmatrix} * \\ 0 \end{pmatrix}$$

in block form.

17: Seeking a power series solution

$$u^a(x) = p^a + p^a_i x^i + p^a_{ij} x^i x^j + p^a_{ijk} x^i x^j x^k + \dots$$

18: Since the derivative of $p^a_i x^i$ is $p^a_i w_a \otimes v^i$, $p^a_i x^i$ is a solution iff $p^a_i w_a \otimes v^i \in A$.

19: If p^a_{ij} is symmetric in i and j, then the derivative of $p^a_{ij} x^i x^j$ is $2p^a_{ij} x^i w_a \otimes v^j$. So $p^a_{ij} x^i x^j$ is a solution on an open set iff

$$x^i (p^a_{ij} w_a \otimes v^j) \in A$$

for all x on an open set. Partial differentiation with respect to x^i gives the necessary condition $p^a_{ij} w_a \otimes v^j \in A$ which is also clearly sufficient for $p^a_{ij} x^i x^j$ to be a solution.

20: The function x^i on V is just contraction with v^i, so for p_{ij} symmetric,

$$x^i (p^a_{ij} w_a \otimes v^j) \in A \Longrightarrow p^a_{ij} w_a \otimes v^i \otimes v^j \in A \otimes V^*.$$

Also $p^a_{ij} w_a \otimes v^i \otimes v^j \in W \otimes S^2(V^*)$ since that is equivalent to symmetry in i and j.

21: The first prolongation $A^{(1)}$ is defined by

$$A^{(1)} = (A \otimes V^*) \cap \left(W \otimes S^2(V^*)\right).$$

So $p^a_{ij} w_a \otimes v^i \otimes v^j \in A^{(1)}$ does imply $x^i (p^a_{ij} w_a \otimes v^j) \in A$ and the symmetric polynomial $p^a_{ij} x^i x^j$ is a solution.

22: The second derivative of $p^a_{ij} x^i x^j$ is $2p^a_{ij} w_a \otimes v^i \otimes v^j$ $inW \otimes S^2(V^*)$. In fact $p^a_{ij} x^i x^j$ is the unique homogeneous W valued quadratic polynomial with this second derivative.

22: If p^a_I is symmetric in the multi-index I with $|I| = k$, then the derivative of $p^a_I x^I$ is $k p^a_{iJ} x^i w_a \otimes v^J$ with $|J| = k - 1$. So $p^a_I x^I$ is a solution on an open set iff

$$x^J (p^a_{iJ} w_a \otimes v^i) \in A$$

for all x on an open set. Partial differentiation with respect to x^J gives the necessary condition $p^a_{iJ} w_a \otimes v^i \in A$ which is also clearly sufficient.

23: The function x^J on V is just contraction with v^J, so for p_I symmetric,

$$x^J (p^a_{iJ} w_a \otimes v^i) \in A \Longrightarrow p^a_{iJ} w_a \otimes v^i \otimes v^J \in A \otimes S^{k-1}(V^*)$$

Also $p^a_I x^I \in W \otimes S^k(V^*)$.

24: The kth prolongation $A^{(k)}$ is defined by

$$A^{(k)} = \left(A \otimes V^{*\otimes k}\right) \cap \left(W \otimes S^{k+1}(V^*)\right).$$

So $p^a_I w_a \otimes v^I \in A^{(k-1)}$ does imply $x^J(p^a_{iJ} w_a \otimes v^J) \in A$ and the symmetric polynomial $p^a_{iJ} x^i x^J$ is a solution.

25: The kth derivative of $p^a_I x^I$ is $k p^a_{iJ} w_a \otimes v^i \otimes v^J \in W \otimes S^k(V^*)$. In fact $p^a_{iJ} x^i x^J$ is the unique homogeneous W valued degree k polynomial with this kth derivative.

26: A tableau of order p is a linear subspace

$$A \subset (W \otimes S^p(V^*)).$$

It determines a homog. cst. coeff. PDE system for functions $V \to W$.

27: Its prolongation is

$$A^{(k)} = \left(A \otimes V^{*\otimes k}\right) \cap \left(W \otimes S^{k+p}(V^*)\right).$$

28: First Example

$$\begin{aligned} A =& \{\left(p^a_1 v^1 + C^a_{2b} p^b_1 v^2 + \ldots + C^a_{nb} p^b_1 v^n\right) \otimes w_a\} \\ =& \{\begin{pmatrix} p_1 & C_2 p_1 & \ldots & C_\rho p_1 & \ldots & C_n p_1 \end{pmatrix}\} \end{aligned}$$

where C_ρ is an $s \times s$ constant matrix.

29: Symbol relations of First Example:

$$\{w^a \otimes v_\rho - C^a_{\rho b} w^b \otimes v_1\}, \quad 2 \le \rho \le n.$$

Differential Equations:

$$\partial_\rho u^a - C^a_{\rho b} \partial_1 u^b = 0.$$

30: Consistency Condition in terms of differentiation:

$$\begin{aligned} \partial_\rho u =& C_\rho \partial_1 u \\ \partial_\sigma u =& C_\sigma \partial_1 u \\ \partial_\rho \partial_\sigma u =& \partial_\sigma \partial_\rho u \\ C_\sigma C_\rho \partial_1^2 u =& C_\rho C_\sigma \partial_1^2 u \end{aligned}$$

for $2 \le \rho, \sigma \le n$.

31: Also determines $\partial_I u$ in terms of $\partial_1^k u$. Namely

$$\partial_I u = C_{i_1} C_{i_2} \ldots C_{i_k} \partial_1^k u.$$

32: Algebraically:

$$\begin{aligned} p^a_i w_a \otimes v^i \in A \Longrightarrow & p_i = C_i p_1 \\ p^a_{ij} w_a \otimes v^i \otimes v^j \in A^{(1)} \Longrightarrow & p_{ij} = C_i C_j p_{11}. \end{aligned}$$

So consistency for all p^a_{11} requires $C_i C_j = C_j C_i$ for $i, j \ge 2$.

33: If all $i_j \geq 2$,

$$p_I^a w_a \otimes v^I \in A^{(k-1)} \Longrightarrow p_I = C_I p_{1\ldots1}$$

and this is $\iff$ when $[C_i, C_j] = 0$ holds.
34: Prop: If $[C_i, C_j] = 0$, $\exists!$ soln with IC

$$u^a(x^1, 0, \ldots, 0) = f^a(x^1).$$

(Majorant argument shows convergence.)
35: There may also be solutions when $[C_i, C_j] \neq 0$.
36: Summary of example 1:

- dim $A^{(1)} \leq s$.
- Largest space of solutions depends on s functions of 1 variable.
- Largest space of solutions arises exactly when dim $A^{(1)} = s$.

37: Second Example (n=3):

$$\begin{aligned} A &= \begin{pmatrix} p_1 & p_2 & Fp_1 + Gq_1 + Hp_2 \\ q_1 & Cp_1 + Dq_1 + Ep_2 & Ip_1 + Jq_1 + Kp_2 \end{pmatrix} \\ &= \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \end{aligned}$$

with $p_i \in R^k$, $q_i \in R^{s-k}$.
38: $1 \leq \lambda, \mu \leq k$, $k+1 \leq \xi, \eta \leq s$. Note e.g.

$$(Cp_1)^\xi = C^\xi_\lambda p_1^\lambda.$$

39: Special case with $A = \ldots = K = 0$ is

$$A = \begin{pmatrix} p_1 & p_2 & 0 \\ q_1 & 0 & 0 \end{pmatrix}$$

corresponding to the equations

$$\begin{aligned} \partial_2 u^\xi &= 0 \\ \partial_3 u^a &= 0 \end{aligned}$$

with solution

$$\begin{aligned} u^\xi &= f^\xi(x^1) \\ u^\lambda &= f^s(x^1, x^2). \end{aligned}$$

(Referred to here as the simplified example 2.)
40: In the simplified example 2, dim $A^{(1)} = s + 2k$ because

$$\begin{aligned} A =& \{w_\lambda \otimes \left(p_1^\lambda v^1 + p_2^\lambda v^2\right) \\ &+ w_\xi \otimes \left(q_1^\xi v^1\right)\} \end{aligned}$$

$$\begin{aligned}A^{(1)} =&\{w_\lambda \otimes \left(p^\lambda_{11} v^1 \otimes v^1 + p^\lambda_{12} v^1 \circ v^2 + p^\lambda_{22} v^2 \otimes v^2\right)\\ &+ w_\xi \otimes \left(q^\xi_{11} v^1 \otimes v^1\right)\}\end{aligned}$$

where $\alpha \circ \beta = \alpha \otimes \beta + \beta \otimes \alpha$.

41: In the simplified example 2,

$$\begin{aligned}A^{(l)} =&\{w_\lambda \otimes \left(p^\lambda_I (v^1)^{\otimes l+1} + p^\lambda_I (v^1)^{\otimes j} \circ (v^2)^{\otimes l+1-j} + p^\lambda_I (v^2)^{\otimes l+1}\right)\\ &+ w_\xi \otimes \left(q^\xi_I (v^1)^{\otimes l+1}\right)\}\end{aligned}$$

where $\alpha \circ \beta = \alpha \otimes \beta + \beta \otimes \alpha$.

42: In the full example 2,

$$\begin{aligned}A = \{&w_\lambda \otimes \left(p^\lambda_1 v^1 + (Fp_1)^\lambda v^3\right) + w_\xi \otimes \left((Cp_1)^\xi v^2 + (Ip_1)^\xi v^3\right) +\\ &w_\lambda \otimes \left(p^\lambda_2 v^2 + (Hp_2)^\lambda v^3\right) + w_\xi \otimes \left((Ep_2)^\xi v^2 + (Kp_2)^\xi v^3\right) +\\ &w_\lambda \otimes \left((Gq_1)^\lambda v^3\right) + w_\xi \otimes \left(q^\xi_1 v^1 + (Dq_1)^\xi v^2 + (Jq_1)^\xi v^3\right)\}\end{aligned}$$

$$\begin{aligned}A^{(l)} = \{&w_\lambda \otimes \left(p^\lambda_1 v^1 \otimes v^J + (Fp_1)^\lambda v^3 \otimes v^J\right) + w_\xi \otimes \left((Cp_1)^\xi v^2 \otimes v^J + (Ip_1)^\xi v^3 \otimes v^J\right) +\\ &w_\lambda \otimes \left(p^\lambda_2 v^2 \otimes v^J + (Hp_2)^\lambda v^3 \otimes v^J\right) + w_\xi \otimes \left((Ep_2)^\xi v^2 \otimes v^J + (Kp_2)^\xi v^3 \otimes v^J\right) +\\ &w_\lambda \otimes \left((Gq_1)^\lambda v^3 \otimes v^J\right) + w_\xi \otimes \left(q^\xi_1 v^1 \otimes v^J + (Dq_1)^\xi v^2 \otimes v^J + (Jq_1)^\xi v^3 \otimes v^J\right)\}\end{aligned}$$

where $|J| = l$.

43: The above can also be written as

$$\begin{aligned}A = \{ &\left(p_1 \otimes v^1 + (Fp_1) \otimes v^3\right) + \left((Cp_1) \otimes v^2 + (Ip_1) \otimes v^3\right) +\\ &\left(p_2 \otimes v^2 + (Hp_2) \otimes v^3\right) + \left((Ep_2) \otimes v^2 + (Kp_2) \otimes v^3\right) +\\ &\left((Gq_1) \otimes v^3\right) + \left(q_1 \otimes v^1 + (Dq_1) \otimes v^2 + (Jq_1) \otimes v^3\right)\}\end{aligned}$$

44: Set

$$\tilde{p}_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix}, \quad \tilde{p}_{iJ} = \begin{pmatrix} p_{iJ} \\ q_{iJ} \end{pmatrix}.$$

Then for symmetric $\tilde{p}$,

$$\tilde{p}^a_{iJ} w_a \otimes v^{iJ} \in A^{(l)} \iff \forall J, \ \ \tilde{p}^a_{iJ} w_a \otimes v^i \in A.$$

Here that means

$$\begin{aligned}q_{2J} &= Cp_{1J} + Dq_{1J} + Ep_{2J}\\ q_{3J} &= Ip_{1J} + Jq_{1J} + Kp_{2J}\\ p_{3J} &= Fp_{1J} + Gq_{1J} + Hp_{2J}\end{aligned}$$

just as differentiating equations like $q_2 = Cp_1 + Dq_1 + Ep_2$ by ∂_J would give.

45: Define (ab) the normal height $/p_I/$ or $/q_I/$ of a term to be:

$/p_I/$: The number of indices in I which are 3.

$/q_I/$: The number of indices in I which are 2 or 3.

The above system says

$$
\begin{aligned}
q_{21J} &= Cp_{11J} + Dq_{11J} + Ep_{21J} \\
q_{22J} &= Cp_{12J} + Dq_{12J} + Ep_{22J} \\
&= Cp_{12J} + D(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Ep_{22J} \\
q_{23J} &= Cp_{13J} + Dq_{13J} + Ep_{23J} \\
&= C(Fp_{11J} + Gq_{11J} + Hp_{21J}) + D(Ip_{11J} + Jq_{11J} + Kp_{21J}) \\
&\quad + E(Fp_{12J} + Gq_{12J} + Hp_{22J})) \\
&= C(Fp_{11J} + Gq_{11J} + Hp_{21J}) + D(Ip_{11J} + Jq_{11J} + Kp_{21J}) \\
&\quad + E(Fp_{12J} + G(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Hp_{22J})) \\
q_{32J} &= Ip_{12J} + Jq_{12J} + Kp_{22J} \\
&= Ip_{12J} + J(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Kp_{22J} \\
q_{31J} &= Ip_{11J} + Jq_{11J} + Kp_{21J} \\
q_{33J} &= Ip_{13J} + Jq_{13J} + Kp_{23J} \\
&= I(Fp_{11J} + Gq_{11J} + Hp_{21J}) + J(Ip_{11J} + Jq_{11J} + Kp_{21J}) \\
&\quad + K(Fp_{12J} + Gq_{12J} + Hp_{22J}) \\
&= I(Fp_{11J} + Gq_{11J} + Hp_{21J}) + J(Ip_{11J} + Jq_{11J} + Kp_{21J}) \\
&\quad + K(Fp_{12J} + G(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Hp_{22J}) \\
p_{31J} &= Fp_{11J} + Gq_{11J} + Hp_{21J} \\
p_{32J} &= Fp_{12J} + Gq_{12J} + Hp_{22J} \\
&= Fp_{12J} + G(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Hp_{22J} \\
p_{33J} &= Fp_{13J} + Gq_{13J} + Hp_{23J} \\
&= F(Fp_{11J} + Gq_{11J} + Hp_{21J}) + G(Ip_{11J} + Jq_{11J} + Kp_{21J}) \\
&\quad + H(Fp_{12J} + G(Cp_{11J} + Dq_{11J} + Ep_{21J}) + Hp_{22J})
\end{aligned}
$$

46: Comparing q_{23J} and q_{32J}, we see as long as the consistency condition

$$
\begin{aligned}
CF + DI + EGC &= JC \\
CH + DK + EFEGE &= I + JE \\
EH &= K \\
CG + DJ + EGD &= JD
\end{aligned}
$$

holds, then the system above uniquely determines every $/p_I/$ or $/q_I/$ of positive normal height in terms of elements of normal height 0.

47: Prop: If the consistency condition holds, $\exists!$ solution to the IVP $\gamma_u(x) \in A$ with IC

$$u^\lambda(x^1, x^2, 0) = \quad f^\lambda(x^1, x^2)$$
$$u^\xi(x^1, 0, 0) = \qquad f^\xi(x^1)$$

as long as f^λ and f^ξ are analytic.

48: So dim $A^{(1)} \leq s + 2k$ with equality if the matrix consistency conditions holds.

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