Flow Polytopes and Degree Sequences

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Joint work with Karola Mészáros
Outline

What are flow polytopes?
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An Interesting Example
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An Interesting Example
Triangulations of Flow Polytopes
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Right-Degree Polynomials
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An Interesting Example
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Right-Degree Polynomials
Further Questions
Flows on a Graph

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Make \( G \) a directed graph with edge orientations induced by the vertex labels, so edges go from smaller to larger vertices.
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View each edge as a pipe, each vertex as a valve with an inflow or outflow amount, and imagine water moving across the graph.
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Make $G$ a directed graph with edge orientations induced by the vertex labels, so edges go from smaller to larger vertices.

View each edge as a pipe, each vertex as a valve with an inflow or outflow amount, and imagine water moving across the graph.

A flow is an assignment $f : E(G) \rightarrow \mathbb{R}_{\geq 0}$ of quantities of fluid to each edge so that fluid is conserved at each vertex.
Flows on a Graph

$K_5$

2 1 -1 0 -2

1 2 3 4 5
Flow polytopes $\mathcal{F}_G(a)$

(Postnikov-Stanley ’05, Baldoni-Vergne ’08) For a graph $G$, on the vertex set $\{1, 2, \ldots, n\}$ and netflow vector $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$,
the flow polytope of $G$ is

$$\mathcal{F}_G(a) := \{ f : E(G) \rightarrow \mathbb{R}_{\geq 0} \mid \text{netflow at vertex } j = a_j \}$$
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The netflow at $j$ is the outflow at $j$ minus the inflow at $j$:

$$\sum_{(j,k) \in E(G), j<k} f((j,k)) - \sum_{(i,j) \in E(G), i<j} f((i,j))$$
**Flow polytopes $\mathcal{F}_G(a)$**

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We call $f \in \mathcal{F}_G(a)$ a **flow** on $G$ with netflow $\mathbf{a}$. 
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We call $f \in \mathcal{F}_G(a)$ a **flow** on $G$ with netflow $a$.

In order for a flow to exist, it is necessary that $\sum_{j=1}^{m} a_j \geq 0$ for each $m < n$, and $\sum_{j=1}^{n} a_j = 0$. 
The flow polytope $\mathcal{F}_{K_5}(1, 0, 0, 0, -1)$

$a, b, c, d, e, f, g, h, i, j \geq 0$
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$1 = a + b + c + d$
The flow polytope $\mathcal{F}_{K_5}(1, 0, 0, 0, -1)$

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\[
\begin{align*}
1 &= a + b + c + d \\
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0 &= h + i - b - e \\
0 &= j - c - f - h \\
-1 &= -(j + i + g + d)
\end{align*}
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$0 = j - c - f - h$

$-1 = -(j + i + g + d)$

If $M_G$ is the incidence matrix of a graph $G$, then another way to view the flow polytope $\mathcal{F}_G(a)$ is

$$\mathcal{F}_G(a) = \left\{ x \in \mathbb{R}_{\geq 0}^{\#E(G)} : M_G x = a \right\}.$$
The number of integer points in a flow polytope $F_G(a)$ is

$$\# \left\{ x \in \mathbb{Z}^{\# E(G)}_{\geq 0} : M_G x = a \right\}.$$  

This is the number of ways to write $a$ as a nonnegative integer combination of the vectors $e_i - e_j$ for $(i, j) \in E(G)$.  

In representation theory, this function of $a$ is called the Kostant partition function.
What is the flow polytope of ...?
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\[ x = x + y + z \]

\[ x, y, z \geq 0 \]
What is the flow polytope of \ldots?

It's \( \Delta^2 \)!
Flow polytopes (in disguise)

The **Chan-Robbins-Yuen polytope**:

\[
\mathcal{CRY}_n := \left\{ (b_{ij}) \in \mathbb{R}^{n \times n}_{\geq 0} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \right\}
\]

= convex hull $n \times n$ permutation matrices

\[
\begin{array}{c}
0 \\
\end{array}
\]
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- \(2^{n-1}\) vertices, dimension \(\binom{n}{2}\)

- A face of the Birkhoff polytope \(B(n)\)

\[ B(n) = \left\{ (b_{ij}) \in \mathbb{R}^{n^2} \mid b_{ij} \geq 0, \ \sum_i b_{ij} = 1, \ \sum_j b_{ij} = 1 \right\} \]

\[ = \text{convex hull all } n \times n \text{ permutation matrices} \]
Flow polytopes (in disguise)

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The normalized volume of \( \mathcal{CRY}_n \) has been shown analytically to be the product of Catalan numbers (Zeilberger, 1999), but no combinatorial proof is known.
$B(n)$ and $CRY_n$  

$B(3)$ and $CRY_3$  

$CRY_3$
From $\mathcal{CRY}_n$ to a flow polytope

$\mathcal{CRY}_n := \{ (b_{ij}) \in \mathbb{R}_{\geq 0}^{n^2} \mid \text{doubly-stochastic matrix, } b_{ij} = 0, i - j \geq 2 \}$

\begin{tabular}{ccc}
 a & b & c \\
■ & d & e \\
■ & f \\
\end{tabular}

$a, b, c, d, e, f \geq 0$
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$$a + b + c = 1$$

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\[
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 a + b + c &= 1 \\
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\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 & a & b & c & d & e & f \\
\hline
\text{■} &  &  &  &  &  &  \\
\text{■} &  &  &  &  &  &  \\
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\[\begin{array}{ccc}
 a & b & c \\
 \blacksquare & d & e \\
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- $\mathcal{CRY}_n$ is equivalent to the flow polytope of the complete graph $K_{n+1}$ with netflow $(1, 0, \ldots, 0, -1)$. 
Triangulations of Flow Polytopes

For certain graphs $H$ with the special netflow vector $\mathbf{a} = (1, 0, \ldots, 0, -1)$, there is a systematic method for triangulating $\mathcal{F}_H(\mathbf{a})$. 
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For any graph $G$ on vertices $\{1, 2, \ldots, n\}$, define a graph $\tilde{G}$ on vertices $\{s, 1, 2, \ldots, n, t\}$ by connecting both $s$ and $t$ to all of the original vertices.
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A Subdivision Lemma

Given a graph $G$, pick edges $(i, j), (j, k)$ in $G$ with $i < j < k$. 
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Given a graph $G$, pick edges $(i, j), (j, k)$ in $G$ with $i < j < k$. Define three new graphs $G_1, G_2, \text{ and } G_3$ on the same vertex set by:

\[
E(G_1) = E(G) \setminus \{(j, k)\} \cup \{(i, k)\}
\]

\[
E(G_2) = E(G) \setminus \{(i, j)\} \cup \{(i, k)\}
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Lemma (Postnikov, Stanley):

For any netflow vector $a$, up to integral equivalence,

$$\mathcal{F}(G) = \mathcal{F}(G_1) \cup \mathcal{F}(G_2), \quad \mathcal{F}(G) \cap \mathcal{F}(G_1) = \emptyset, \text{ and } \mathcal{F}(G_1) \cap \mathcal{F}(G_2) = \mathcal{F}(G_3)$$
Reduction Trees

Given a graph $G$, call the construction of the graphs $G_1$, $G_2$, and $G_3$ a reduction on $G$. 
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To subdivide the flow polytope of $G$, repeatedly perform reductions of $G$. This process is encoded in a reduction tree.
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To subdivide the flow polytope of $G$, repeatedly perform reductions of $G$. This process is encoded in a reduction tree.

To build a reduction tree of $G$ begin with a root node labeled by $G$. Add three children labeled by $G_1$, $G_2$, and $G_3$ for a choice of reduction. Iterate this process until the graphs labeling the leaves of the tree cannot be reduced further.
Reduction Trees
Reduction Trees
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Reduction Trees
Reduction Trees

If $L_1, \ldots, L_m$ are the graphs labeling the leaves, of a reduction tree of $G$ that have the same number of edges as $G$, then the subdivision Lemma implies $\left\{ \mathcal{F}_{L_i} : i = 1, \ldots, m \right\}$ induces a polyhedral subdivision of $\mathcal{F}_{\tilde{G}}$. 
Why the Tilde?

The Subdivision Lemma holds without the tildes, but then the reduction tree only gives a dissection of $F_G$, not necessarily a triangulation.
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With the tildes though...
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If a vertex in a graph with netflow 0 has only one incoming or outgoing edge, contracting that edge yields a graph with an equivalent flow polytope.
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$$L$$

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$$\tilde{L} s \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & -1 \\ \end{array} t \cong \begin{array}{c} 1 \\ -1 \end{array} = \Delta^7$$

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If a vertex with netflow 0 in a graph has only one incoming (outgoing) edge, contracting that edge yields a graph with an equivalent flow polytope.

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In the triangulation, the cells correspond to leaves $L$ in the reduction tree with $\#E(L) = \#E(G)$, and their intersections correspond to the other leaves.
Degree Sequences

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For a graph $G$ on vertices $\{1, 2, \ldots, n\}$, let $\text{outdeg}_G(i)$ be the number of edges $(i, j)$ with $i < j$. The right-degree sequence of $G$ is the vector $(\text{outdeg}_G(1), \ldots, \text{outdeg}_G(n - 1))$. 
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$$(3, 2, 0, 1)$$
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Let $\text{RD}(G)$ denote the multiset of right-degree vectors of leaves in a reduction tree of $G$.
Example Reduction Tree
Example Reduction Tree

\[ RD(G) = \{(3,1,0), (2,1,0), (2,2,0), (2,1,0), (3,1,0), (2,1,1), (2,1,0)\} \]
Theorem (Escobar, Mészáros):
When $G$ is a tree, $RD(G)$ is independent of the choice of reduction tree.
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Theorem (Mészáros, St. Dizier): True for any $G$!
Right-Degree Polynomials

For a graph $G$ on $[n]$, define the right-degree polynomial $\mathcal{R}_G$ of $G$ by:

$$\mathcal{R}_G(t_1, t_2, \ldots t_{n-1}) = \sum_{\alpha \in \text{RD}(G)} (-1)^{\text{codim}(\alpha)} t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_{n-1}^{\alpha_{n-1}}$$

where $\text{codim}(\alpha) = \#E(G) - \sum_{j=1}^{n-1} \alpha_j$.
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where $\text{codim}(\alpha) = \#E(G) - \sum_{j=1}^{n-1} \alpha_j$.

$$RD(G) = \{(4, 0, 1, 0), (4, 0, 1, 0), (3, 0, 2, 0), (3, 0, 1, 1), (3, 0, 1, 0), (3, 0, 1, 0), (3, 0, 1, 0)\}$$

$$R_G(t_1, t_2, t_3, t_4) = 2t_1^4 t_3 + t_1^3 t_2^2 + t_1^3 t_3 t_4 - 3t_1^3 t_3$$
Newton Polytopes

For a polynomial $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_n]$, the Newton polytope is

$$\text{Newton}(f) = \text{Conv} \{ \alpha : c_{\alpha} \neq 0 \}.$$
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(Standard) Permutahedra - polytopes whose vertices are all rearrangements of a list of numbers.
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$$\text{Newton}(f) = \text{Conv} \{ \alpha : c_\alpha \neq 0 \}.$$ 

The Newton polytope of a right-degree polynomial decomposes as a union of generalized permutahedra.

(Standard) Permutahedra - polytopes whose vertices are all rearrangements of a list of numbers.

Generalized Permutahedra - polytopes obtained by deforming a standard permutahedron by parallel translation of the facets.
Further Questions

Goal: Use a good understanding of right-degree polynomials to study other families of polynomials in algebraic combinatorics.
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For certain permutations $\pi \in S_n$, the Grothendieck polynomial $G_\pi$ is a shift of the right-degree polynomial of a tree $T(\pi)$. In particular the Schubert polynomial $S_\pi$ is a shift of the top homogeneous component of $R_{T(\pi)}$. 
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To other classes of polynomials such as key polynomials?
Thanks for listening!