MATH 3560 Groups and Geometry
Prelim 1 SOLUTIONS

Problem 1. (20 points) Complete the following definitions:

(a) A function $f : \mathbb{C} \to \mathbb{C}$ is an *isometry* if ...

$$|f(z) - f(w)| = |z - w| \text{ for every } z, w \in \mathbb{C}.$$

(b) Let X and Y be sets.

(i) A function $f : X \to Y$ is *one-to-one* if ...

whenever $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$, then $x_1 = x_2$,
OR whenever x_1 and x_2 are elements of X such that $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(ii) A function $f : X \to Y$ is *onto* if ...

for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.

(iii) A function $f : X \to Y$ is a *bijection* if ...

it is both one-to-one and onto.

(c) Let $(G, \circ)$ and $(H, *)$ be groups (that is, G is a group under the operation $\circ$ and H is a group under the operation $*$).

(i) A function $\phi : (G, \circ) \to (H, *)$ is a *homomorphism* if ...

for every $g_1, g_2 \in G$, $\phi(g_1 \circ g_2) = \phi(g_1) * \phi(g_2)$.

(ii) A function $\phi : (G, \circ) \to (H, *)$ is an *isomorphism* if ...

it is both a bijection and a homomorphism.

(d) Let $(G, \circ)$ be a group.

(i) A function $\phi : G \to G$ is an *automorphism* if ...

it is an isomorphism.

(ii) The *identity* element of G is the element $e \in G$ such that ...

$ge = eg$ for every element $g \in G$.

Problem 2. (20 points) This problem consists of several unrelated parts.

(a) State a formula, using the coordinate system of your choice, for the isometry M of the plane which reflects every point in the horizontal line $y = 1$.

We recall that reflection across the x-axis is given by $z \mapsto \overline{z}$. We can therefore obtain reflection in the line $y = 1$ by first translating any point of $y = 1$ to the origin, applying complex conjugation, and then translating the point back:

$$M(z) = \overline{(z - i)} + i = \overline{z} + 2i.$$

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = e^{\pi i/4}(\overline{z} + i)$. Write a synthetic description (that is, a description in words) of what f does to a point z in the plane.

f is the composition of three isometries: first, reflection in the x-axis; second, translation by i; and third, rotation by $\pi/4$ around the origin. The composition of the first two isometries is reflection in the line $y = 1/2$, and the composition of a reflection and a rotation is again a reflection, in this case reflection in the line of equation $e^{\pi i/4}(\overline{z} + i) = z$.

(c) State an example of a subgroup of Isom($\mathbb{C}$) of order 5. (You may describe it in words, or use the symbols for isometries that were introduced in class.)

An example of a subgroup of order 5 in Isom($\mathbb{C}$) is given by the rotations by multiples of $2\pi/5$ around the origin. As a set, this group is

$$\{\mathrm{id}_{\mathbb{C}}, R_{2\pi/5}, R_{4\pi/5}, R_{6\pi/5}, R_{8\pi/5}\}.$$

It may also be thought of as the set of functions

$$\{z \mapsto e^{i\theta}z : \theta = 2k\pi/5 \text{ for some integer } k\}.$$

There are lots of other groups of order 5 in Isom($\mathbb{C}$): take the rotations by the same set of angles around any other point.

Problem 3. (20 points) For each of the statements below, write a T in the blank if the statement is true; write an F in the blank if the statement is false. You are not required to give any reasons for your answer. Illegible letters will be graded as incorrect.

_ F _ The line L is parallel to the line L', where

$$L = \{z \in \mathbb{C} \mid z = i + t(1+2i), t \in \mathbb{R}\}$$

$$L' = \left\{w \in \mathbb{C} \mid w = -\frac{1}{2} + s(-3+6i), s \in \mathbb{R}\right\}.$$

_ F _ The line L is equal to the line L', where

$$L = \{z \in \mathbb{C} \mid z = i + t(1+2i), t \in \mathbb{R}\}$$

$$L' = \{w \in \mathbb{C} \mid w = 2 + s(-3-6i), s \in \mathbb{R}\}.$$

_ F _ The set of all rational numbers (including $0 = 0/1$) in lowest terms with denominators equal to 1, 2 or 3 is a group under addition.

_ T _ The set of all rational numbers (including $0 = 0/1$) in lowest terms with denominators equal to 1 or 2 is a group under addition.

_ F _ Every bijection $f : \mathbb{C} \to \mathbb{C}$ is an isometry.

_ T _ Every isometry $f : \mathbb{C} \to \mathbb{C}$ is a bijection.

_ T _ Let T and T' be any two translations of the plane. Then

$$T \circ T' = T' \circ T.$$

_ T _ Let R and R' be any two rotations of the plane about the origin. Then

$$R \circ R' = R' \circ R.$$

_ F _ Let M and M' be any two reflections of the plane in lines through the origin. Then

$$M \circ M' = M' \circ M.$$

_ F _ Let M and M' be any two reflections of the plane in lines (not necessarily through the origin). Then

$$M \circ M' = M' \circ M.$$

Problem 4. (20 points) This problem consists of two unrelated parts, and continues on the next page.

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = \overline{z}$. Prove that f is an automorphism of the additive group $(\mathbb{C}, +)$. You may use properties of complex conjugation that were established in class and in homework.

> We know that $\overline{z+w} = \overline{z} + \overline{w}$ for any $z, w \in \mathbb{C}$. This shows that f is a homomorphism from $(\mathbb{C}, +)$ to $(\mathbb{C}, +)$. It is also a bijection, because it is an isometry of $\mathbb{C}$. Because f is both a bijection and a homomorphism, it is an isomorphism, and because it sends $(\mathbb{C}, +)$ to itself, it is by definition an automorphism.

(b) Let M_2 be the set of 2×2 matrices with real entries. Consider the binary operation $\diamond$ on M_2 defined for $A, B \in M_2$ by

$$A \diamond B = \frac{1}{2}(AB + BA).$$

That is, $A \diamond B$ is the element of M_2 obtained by multiplying every entry in the matrix $AB + BA$ by the constant $\frac{1}{2}$.

(i) Show that $A \diamond B = B \diamond A$ for all $A, B \in M_2$.

> For every $A, B \in M_2$,
>
> $$A \diamond B = \frac{1}{2}(AB + BA) = \frac{1}{2}(BA + AB) = B \diamond A.$$
>
> We have used the fact that addition of matrices is commutative, which follows from the commutativity of addition in $\mathbb{R}$ and the componentwise definition of matrix addition.

(ii) Find an identity element of M_2 for the binary operation $\diamond$.

Let I be the identity 2×2 matrix. Because $AI = IA = A$,

$$A \diamond I = \frac{1}{2}(AI + IA) = \frac{1}{2}(A + A) = \frac{1}{2}(2A) = A.$$

The fact that $I \diamond A = A$ follows from a similar computation, or from part (i). Therefore I is an identity element of M_2 for $\diamond$.

(iii) Is M_2 a group under the operation $\diamond$? Give reasons for your answer.

M_2 is *not* a group under $\diamond$. Here are two possible reasons that could be given:

- $\diamond$ is not associative: given $A, B, C \in M_2$,

  $$(A \diamond B) \diamond C = \frac{1}{2}(AB + BA) \diamond C = \frac{1}{4}(ABC + BAC + CAB + CBA)$$

  but

  $$A \diamond (B \diamond C) = A \diamond \left(\frac{1}{2}(BC + CB)\right) = \frac{1}{4}(ABC + ACB + BCA + CBA),$$

  and these are not equal in general.
- Not every element has an inverse: consider $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We have

  $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \diamond \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} a & b/2 \\ c/2 & 0 \end{pmatrix},$$

  and there are no values of a, b, c, d that make this equal the identity I.

Problem 5. (20 points) Let G be a group. Recall that the *order* of an element $g \in G$ is the smallest positive integer n such that $g^n = e_G$, where e_G is the identity in G. If there is no such integer n, then we say that g has infinite order.

(a) Suppose that G and H are groups and that $\phi : G \to H$ is an isomorphism.

(i) Suppose $g \in G$ has finite order n. Prove that $\phi(g) \in H$ has order n.

By the definition and properties of a homomorphism,

$$\phi(g)^n = \phi(g^n) = \phi(e_G) = e_H.$$

Therefore $\phi(g)$ has finite order, and it is at most n. Suppose the order of $\phi(g)$ were smaller than n, say $N < n$. Because ϕ is an isomorphism, it has an inverse ϕ^{-1} which is also an isomorphism. We would have:

$$g^N = \phi^{-1}(\phi(g))^N = \phi^{-1}(\phi(g)^N) = \phi^{-1}(e_H) = e_G,$$

which contradicts the claim that g has order n. Therefore the order of $\phi(g)$ is n.

(ii) Suppose $g \in G$ has infinite order. Prove that $\phi(g) \in H$ has infinite order.

Again, because ϕ is an isomorphism, so is ϕ^{-1}. By the argument in part (a), if $\phi(g)$ has some finite order N, then $g = \phi^{-1}(\phi(g))$ would also have to have order N. This contradicts the assumption that g has infinite order, and therefore $\phi(g)$ must have infinite order.

(b) Find two elements f and g of Isom($\mathbb{C}$) such that f and g both have finite order, but their composition $f \circ g$ has infinite order. Justify your answer.

Here are two solutions; many others are possible.

- Let M_0 be reflection in the x-axis, and let M_1 be reflection in the line that forms an angle of 1 radian with the x-axis. Both of these have finite order, because any reflection has order 2. The composition $M_1 M_0$ is rotation by 2 radians around the origin. This element has infinite order, because π is irrational and therefore no non-zero multiple of 2 can equal a multiple of π.
- Let $R_{0,\pi}$ be rotation by π around the origin, and let $R_{1,\pi}$ be rotation by π around $1 = (1, 0)$. Both of these have order 2. The composition $R_{1,\pi} R_{0,\pi}$ is translation by 2 (i.e., the vector $(2, 0)$). This element has infinite order, because every translation by a non-zero vector has infinite order.