# MATH 105: Prelim 3 Solutions

**Problem 1.** Sort the weights from least to greatest:

$$2, 3, 3, 4, 5, 5, 5, 6, 6, 6, 6, 8, 9, 9$$

**(a)** The sum of the values is $\sum x = 2 + 3 + 3 + \cdots + 9 = 77$, and there are $n = 14$ values, so the mean is $\bar{x} = \frac{\sum x}{n} = \frac{77}{14} = 5.5$. Since there are 14 values, the median is the average of the 7th and 8th values: $\tilde{x} = \frac{5+6}{2} = 5.5$.

**(b)** The sum of the squared values is $\sum x^2 = 2^2 + 3^2 + 3^2 + \cdots + 9^2 = 483$. Hence the standard deviation is

$$\begin{aligned} s^2 &= \sqrt{\frac{\sum x^2 - n(\bar{x})^2}{n-1}} \\ &= \sqrt{\frac{483 - 14(5.5)^2}{13}} \\ &= \sqrt{4.567923077} \approx 2.14. \end{aligned}$$

**(c)** One standard deviation below the mean is $5.5 - 2.14 = 3.36$ and one standard deviation above the mean is $5.5 + 2.14 = 7.64$. There are 8 gerbils that fall between these values.

**(d)** For a normal distribution, there is a 68.29% chance of falling within one standard deviation of the mean (this can be checked by finding the area between $-1$ and 1 under the standard normal curve). Hence if our group of 14 gerbils were typical, we would expect $14\,(.6829) = 9.5606 \approx 9.6$ of them to fall within one standard deviation of the mean.

**Problem 2.**

**(a)** There are $\binom{5}{2} = 10$ equally likely outcomes in this experiment. We list each possibility, along with the sum of the values on the two balls.

| outcome | 1,2 | 1,3 | 1,4 | 1,5 | 2,3 | 2,4 | 2,5 | 3,4 | 3,5 | 4,5 |
|---|---|---|---|---|---|---|---|---|---|---|
| sum | 3 | 4 | 5 | 6 | 5 | 6 | 7 | 7 | 8 | 9 |

Let $x$ be the profit of player $A$. Notice that $x$ can take on seven different values: $3, -4, 5, -6, 7, -8, 9$. Consulting our table above, we find the distribution of this random variable.

| $x$ | 3 | $-4$ | 5 | $-6$ | 7 | $-8$ | 9 |
|---|---|---|---|---|---|---|---|
| $P(x)$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{2}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ |

**(b)** The expected value of $x$ is

$$\begin{aligned} E(x) &= 3(\frac{1}{10}) - 4(\frac{1}{10}) + 5(\frac{2}{10}) - 6(\frac{2}{10}) + 7(\frac{2}{10}) - 8(\frac{1}{10}) + 9(\frac{1}{10}) \\ &= \frac{3 - 4 + 10 - 12 + 14 - 8 + 9}{10} \\ &= \frac{12}{10} = \$1.20. \end{aligned}$$

Hence, on average, player $A$ can expect to win \$1.20 by playing this game (and obviously player $B$ will lose \$1.20 on average). We should choose to play as $A$.

**Problem 3.** Let "success" be getting a defective glass. Then assuming that the defectiveness of different glasses is independent, this is a binomial experiment with $n = 1000$ and $p = .05$. Let $x$ be the number of defective glasses. Since $np = 50$ and $n(1-p) = 950$ are both greater than 5, it is safe to approximate $x$ by a normal random variable with $\mu = np = 50$ and $\sigma = \sqrt{np(1-p)} \approx 6.89$.

**(a)** $P(\text{at least 55 are defective})$

$$\begin{aligned} &\approx P(x > 54.5) \\ &= P\left(\frac{x - 50}{6.89} > \frac{54.5 - 50}{6.89}\right) \\ &= P(z > .65) \\ &= 1 - .7422 \\ &= .2578 = 25.78\% \end{aligned}$$

**(b)** $P(\text{between 50 and 55 are defective})$

$$\begin{aligned} &\approx P(49.5 < x < 55.5) \\ &= P\left(\frac{49.5 - 50}{6.89} < \frac{x - 50}{6.89} < \frac{55.5 - 50}{6.89}\right) \\ &= P(-.07 < z < .80) \\ &= P(z < .80) - P(z < -.07) \\ &= .7881 - .4721 \\ &= .3160 = 31.6\% \end{aligned}$$

**Problem 4.** Let $x$ be the wingspan of one of these birds. Then $x$ is a normal random variable with $\mu = 12.1$ and $\sigma = 3$.

**(a)**

$$
\begin{aligned}
P(x > 14) &= P\left(\frac{x - 12.1}{3} > \frac{14 - 12.1}{3}\right) \\
&= P(z > .63) \\
&= 1 - .7357 \\
&= .2643 = 26.43\%
\end{aligned}
$$

**(b)**

$$
\begin{aligned}
P(11 < x < 13) &= P\left(\frac{11 - 12.1}{3} < \frac{x - 12.1}{3} < \frac{13 - 12.1}{3}\right) \\
&= P(-.37 < z < .30) \\
&= P(z < .30) - P(z < -.37) \\
&= .6179 - .3557 \\
&= .2622 = 26.22\%
\end{aligned}
$$

**(c)** By reverse look-up, we find that there is approximately 30% of the area to the right of $z = .52$ under the standard normal distribution. This indicates that if $x$ is a normal random variable with mean $\mu$ and standard deviation $\sigma$, there will be 30% of the area to the right of $x = \mu + (.52)\sigma$. Indeed, we have

$$
\begin{aligned}
z &= \frac{x - \mu}{\sigma} \\
\sigma z &= x - \mu \\
x &= \mu + \sigma z.
\end{aligned}
$$

In this case, $x = 12.1 + (.52)3 = 13.66$. Therefore, 30% of these kind of birds have a wingspan greater than 13.66 inches.

**Problem 5.**

**(a)** This is a binomial experiment with $n = 7$ and $p = \frac{3}{4}$. The probability that Henri wakes up with a hangover exactly three mornings in these seven days is

$$\binom{7}{3}(.75)^3(.25)^4 \approx .0577 = 5.77\%$$

**(b)** The probability that Henri wakes up with a hangover at most four mornings in these seven days is

$$
\begin{aligned}
&1 - \left(\binom{7}{5}(.75)^5(.25)^2 + \binom{7}{6}(.75)^6(.25)^1 + \binom{7}{7}(.75)^7(.25)^0\right) \\
&\approx 1 - (.3115 + .3115 + .1335) \\
&= .2435 = 24.35\%
\end{aligned}
$$

**(c)** Let $p = .2435$. Let "success" be "Henri wakes up with a hangover at most four mornings this week". The probability of success in all three of the next three weeks is

$$\binom{3}{3}(.2435)^3(.7565)^0 = (.2435)^3 \approx .0014 = 1.4\%.$$

**(d)** The probability of success in exactly 9 of the 52 weeks of the year is

$$\binom{52}{9}(.2435)^9(.7565)^{43} \approx 0.068 = 6.8\%.$$

Luckily, this calculation is just barely feasible if your calculator can handle ten digits. If not, the unevaluated expression is sufficient.