Math 105, Fall 2004 – Solutions to Final Exam

1. In order to solve the system we construct the associated matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 5 \end{array}\right].$$

Using the row-echelon method this matrix can be transformed into

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The solution $x = 1, y = 0, z = 2$ is obtained by back substitution.

2. If you draw a diagram corresponding to this chain, you will see that if there is any justice in the world the answer better be $[1/3, 1/3, 1/3]$. You can check directly that this is indeed the correct answer.

If you don't notice that there is a lot of symmetry then you can follow the standard method. That is, you need to find a vector $v = [v_1, v_2, v_3]$ satisfying

- $vP = v$
- $v_1 + v_2 + v_3 = 1.$

The first bullet leads us to consider the following system of equations

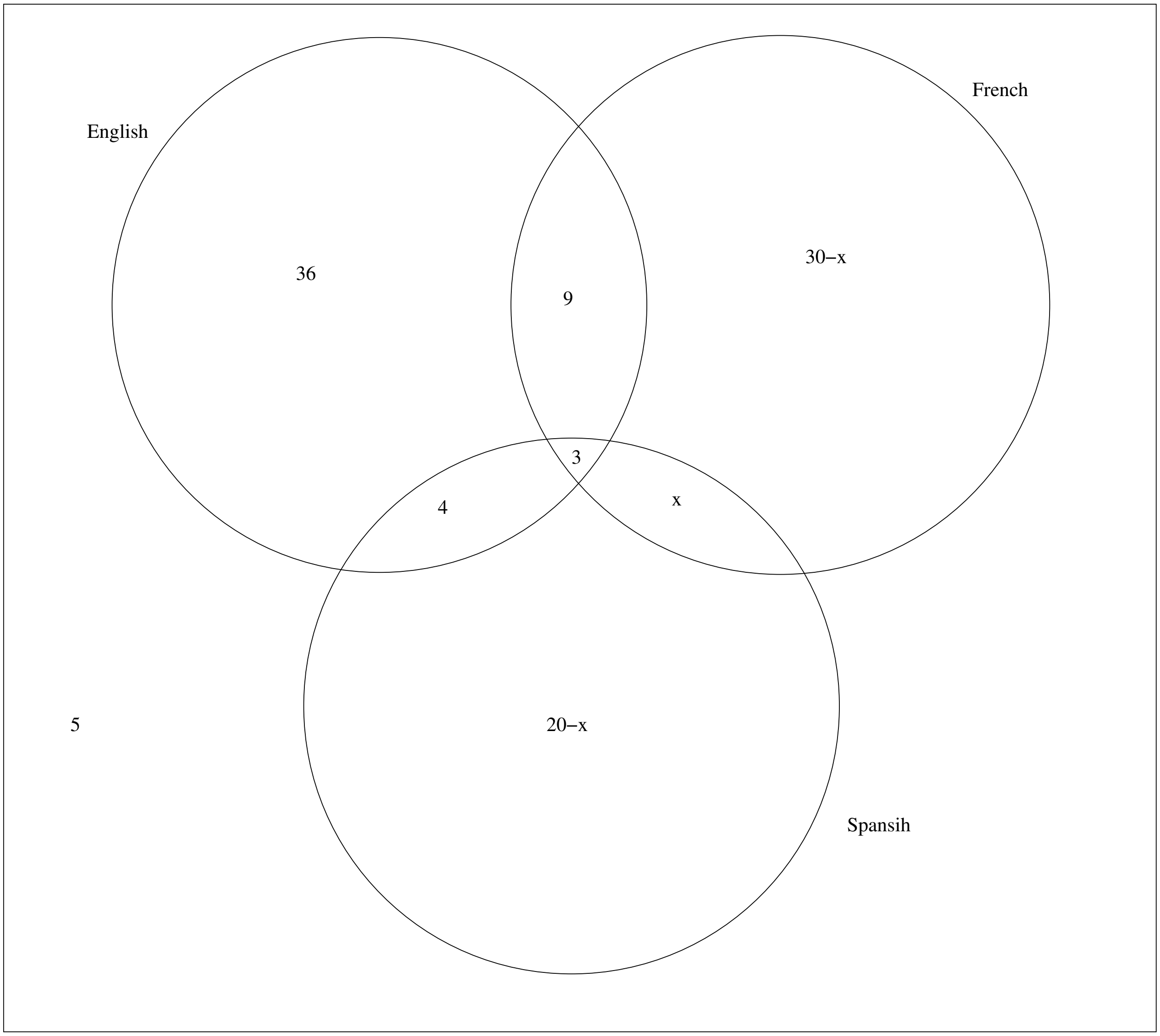
$$\begin{aligned} -v_1 + 1/2v_2 + 1/2v_3 &= 0 \\ 1/2v_1 - v_2 + 1/2v_3 &= 0 \\ 1/2v_1 + 1/2v_2 - v_3 &= 0. \end{aligned}$$

Applying the row-echelon method we find the solutions to be $v = [t, t, t]$, where t is any number.

Now, using the second bullet we see that $t = 1/3$, and hence, the equilibrium vector is seen to be

$$[1/3, 1/3, 1/3].$$

3. Draw and fill in a Venn diagram as follows:



(a) From the diagram, we have the following equation:

$$5 + 36 + 9 + 3 + 4 + x + (30 - x) + (20 - x) = 100$$

Solving for x, we get $x = 7$. Hence $x + 3 = 10$ people speak both French and Spanish.

(b) From the diagram, we see that $20 - x = 13$ people Spanish, but neither French nor English.

4.

(a) **(i)** This is the number of ways of unordered choosing pairs from five people, i.e.

$$\binom{5}{2} = 10\,.$$

(ii) Antonio must play every other player exactly once, and there four other players. Hence Antonio must play four games.

(b) This is a binomial distribution with $n = 4$ and $p = \frac{1}{3}$. To see that $p = \frac{1}{3}$, let R_A, P_A, S_A be the events that Antonio shows rock, paper or scissors respectively, and let R_O P_O, S_O be the events that his opponets shows rock, paper or scissiors, respectively. Note that since each player shows rock, paper or scissors with equal probability, we have

$$P(R_A) = P(P_A) = P(S_A) = P(R_O) = P(P_O) = P(S_O) = \frac{1}{3}\,.$$

Furthermore, since Antonio's actions are independent of that of his opponent, we have

$$\begin{aligned} p &= P(\text{Antonio wins}) \\ &= P(R_A\,|\,S_O)\cdot P(S_O) + P(P_A\,|\,R_O)\cdot P(R_O) + P(S_A\,|\,P_O)\cdot P(P_O) \\ &= P(R_A)\cdot P(S_O) + P(P_A)\cdot P(R_O) + P(S_A)\cdot P(P_O) \\ &= \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{3}\,. \end{aligned}$$

Using the formula for binomial probability, the probability that Antonio wins exactly two games is

$$\binom{4}{2}\left(\frac{1}{3}\right)^2\left(1-\frac{1}{3}\right)^2 = \frac{8}{27}\,.$$

(c) In this scenario, the event that Antonio wins against Drew is the same as Antonio showing paper. Hence the probability that Antonio wins against Drew is $\frac{1}{3}$.

5.

(a) We are given $P(T|S') = .01$ and $P(T'|S) = .02$. Hence $P(T'|S') = 1 - P(T|S') = .99$ and $P(T|S) = 1 - P(T'|S) = .98$.

(b) We have

$$\begin{aligned} P(T) &= P(S)P(T|S) + P(S')P(T|S') = .01(.98) + .99(.01) = .0197, \\ P(T') &= P(S)P(T'|S) + P(S')P(T'|S') = .01(.02) + .99(.99) = .9803. \end{aligned}$$

So 1.97% of the population will test positive, and 98.03% of the population will test negative for scrumpox.

(c) Using conditional probability (or Bayes' Theorem), we have

$$\begin{aligned} P(S|T) &= \frac{P(S\cap T)}{P(T)} = \frac{P(S)P(T|S)}{P(T)} \\ &= \frac{.01(.98)}{.0197} \\ &\approx .4975. \end{aligned}$$

Given that a person tests positive for scrumpox, there is a 49.75% chance they actually have the disease.

(d) Now, we assume that $P(S) = .1$ and $P(S') = .9$. Using Bayes' Theorem, we have

$$\begin{aligned} P(S|T) &= \frac{P(S)P(T|S)}{P(S)P(T|S) + P(S')P(T|S')} \\ &= \frac{.1(.98)}{.1(.98) + .9(.01)} \\ &\approx .9159. \end{aligned}$$

In this case, about 91.51% of people who test positive actually have the disease.

6.

(a)

$$\frac{109,135}{1,000,000} \approx 10.9\% \qquad \frac{879,171}{1,000,000} \approx 87.9\%$$

(b)

x	49995	395	15	0	-5
$P(x)$	.000002	.000366	.011326	.109135	.879171

(c)

$$\begin{aligned} E(x) &= 49995(.000002) + 395(.000366) + 15(.011326) + 0(.109135) - 5(.879171) \\ &= -3.981405 \approx -\$3.98 \end{aligned}$$

7.

(a) $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 = 16,807$

(b) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$

(c) $2 \cdot 2 \cdot 7 \cdot 7 \cdot 7 = 1,372$

(d) $2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 = 120$

(e)

$$1 - \frac{2,520}{16,807} = 1 - .1499 = .8501 \approx 85\%$$

8.

(a)

(b) $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(c) It is $\frac{5}{16}$ because $(1\ 0)P^2 = (\frac{11}{16}\ \frac{5}{16})$.

(d) Yes, because its entries are all positive.

(e) Solve the system

$$\begin{aligned} x + y &= 1 \\ \frac{3}{4}x + \frac{1}{2}y &= x \\ \frac{1}{4}x + \frac{1}{2}y &= y. \end{aligned}$$

The solution is $x = \frac{2}{3}$ and $y\frac{1}{3}$.

9.

(a)

(b) Ordering the states $2, 4, 6, 1, 3, 5$ we get the matrix: $P = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0. \end{array}\right)$

(c) $F = \begin{pmatrix} \frac{27}{19} & \frac{18}{19} & \frac{12}{19} \\ \frac{12}{19} & \frac{27}{19} & \frac{18}{19} \\ \frac{18}{19} & \frac{12}{19} & \frac{27}{19} \end{pmatrix}$ $\qquad FR = \begin{pmatrix} \frac{9}{19} & \frac{6}{19} & \frac{4}{19} \\ \frac{4}{19} & \frac{9}{19} & \frac{6}{19} \\ \frac{6}{19} & \frac{4}{19} & \frac{9}{19} \end{pmatrix}$

(d) $\frac{27}{19}$

(e) $\frac{27}{19} + \frac{18}{19} + \frac{12}{19} = 3$

(f) $\frac{9}{19}$