Make sure that this examination has 10 numbered pages

Cornell University
Final Examination
December 9, 2004

Mathematics 105

Finite Mathematics for the Life and Social Sciences

Closed Book Examination Time: 2.5 hours

Name: ______________________________

Instructor: ____________ **Section:** ______

Read all of the following information before starting the exam.

*You have **2.5** hours to complete this exam. Please read all instructions carefully, and check your answers. Show all work neatly and in order, and clearly indicate your final answers. Answers must be justified whenever possible in order to earn full credit. **Unless otherwise specified, no credit will be given for unsupported answers, even if your final answer is correct.** Points will be deducted for incoherent, incorrect, and/or irrelevant statements.*

Calculators are permitted; however, you will still need to show your work in any problem involving systems of linear equations, matrix multiplication, or matrix inverses. There is an attached page of formulæ. There are no other aids allowed.

*This test has **10** numbered pages with **9** questions totalling **150** points. Before you hand in your exam, make sure you have all of the pages.*

DO NOT WRITE BELOW THIS LINE

Problem 1 ______	Problem 4 ______	Problem 7 ______
Problem 2 ______	Problem 5 ______	Problem 8 ______
Problem 3 ______	Problem 6 ______	Problem 9 ______

TOTAL ______

Do not detach this page from the exam booklet

Bayes' Theorem

$$P(F_i|E) = \frac{P(F_i)P(E|F_i)}{P(F_1)P(E|F_1) + P(F_2)P(E|F_2) + \cdots + P(F_n)P(E|F_n)}$$

Standard Deviation
The standard deviation of the n numbers $x_1, x_2, x_3, \ldots, x_n$, with mean $\overline{x}$, is

$$s = \sqrt{\frac{\sum x_i^2 - n(\overline{x})^2}{n-1}}.$$

Binomial Distribution
For the binomial distribution, the mean and standard deviation are given by

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{np(1-p)}$$

where n is the number of trials and p is the probability of success on a single trial.

If p is the probability of success and n the number of trials, then the probability of exactly x successes is

$$\binom{n}{x} p^x (1-p)^{n-x}.$$

Expected Value
Suppose the random variable x can take on the n values $x_1, x_2, \ldots, x_n$. Also, suppose the probabilities that these values occur are, respectively, $p_1, p_2, \ldots, p_n$. Then the *expected value* of the random variable is

$$E(x) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n.$$

Absorbing Markov Chains
If the transition matrix for an absorbing Markov chain is

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right]$$

then the associated fundamental matrix is $F = (I - Q)^{-1}$.

1. (*15 points*) Solve the following system of equations. (Show your work.)

$$
\begin{aligned}
x + y + z &= 3 \\
2x + y - z &= 0 \\
x + 2z &= 5
\end{aligned}
$$

2. (*15 points*) The following Markov chain is a *regular* Markov chain. (You do not need to explain why the chain is regular.)

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}.$$

Find its equilibrium vector v. (Show your work.)

3. (*15 points*) Out of a group of 100 people, 52 speak English, 42 speak French and 27 speak Spanish. Furthermore, 5 people speak none of these languages, 3 speak all three, 7 speak English and Spanish, and 12 speak English and French.

(a) How many people speak both French and Spanish?

(b) How many people speak Spanish, but neither French nor English?

4. (*15 points*) Antonio, Drew, Henri, José and Laurent play a tournament of "rock, paper, scissors". This is a game in which there are two players, each of whom reveals either rock, paper or scissors. Rock wins over scissors, scissors wins over paper and paper wins over rock. If both players reveal the same item, the game is a draw. In the tournament, every person plays every other person exactly once.

(a) **(i)** How many games are there in the tournament?

(ii) How many games does Antonio play in the tournament?

(b) Suppose each player randomly shows rock, paper or scissors with equal probability in any given game. What is the probability that Antonio wins exactly two games in the tournament?

(c) Suppose now that Drew always shows rock. If Antonio randomly shows rock, paper or scissors with equal probability, what is the probability that Antonio wins against Drew?

5. (*15 points*) Dr. Banjo Zuckerman has just come up with a new diagnostic test for the disease scrumpox. He has determined that the test has a 1% false positive rate (that is, the test returns positive for 1% of people who **don't** have scrumpox), and the test has a 2% false negative rate (that is, the test returns negative for 2% of people who **do** have scrumpox). Suppose that 1 in 100 people in the population has scrumpox.

(a) Suppose Dr. Zuckerman tests a random person for scrumpox. Let T be the event "tests positive" and let S be the event "has scrumpox". Find $P(T|S)$ and $P(T'|S')$.

(b) What percent of the population will test positive for scrumpox? What percent will test negative for scrumpox?

(c) Given that a person tests positive for scrumpox, what is the chance that they actually have the disease?

(d) Now suppose that 1 in 10 people in the population has scrumpox. Considering this, what is the chance that a person who tests positive actually has the disease?

6. (*15 points*) In a certain lottery game, the following prizes are awarded each week.

Prize	Number of Winners
\$50,000	2
\$400	366
\$20	11,326
\$5	109,135

Suppose that 1,000,000 tickets are sold for each drawing.

(a) If you buy one of these tickets, what is your chance of winning the \$5 prize? What is your chance of not getting a prize?

(b) Suppose the tickets for this game cost \$5, and you buy one ticket. Let x be the value of your profit from playing this game (remember, profit can be negative). Write down a table giving the distribution of this random variable.

(c) What is the expected value of x?

7. (*20 points*) Suppose I have five friends named Alan, Bill, Chuck, Dan and Ed, and I call each of them on the phone once per week. In order not to forget, I make a schedule assigning each friend a day of the week for his phone call.

(a) In how many ways can I make this assignment, if I am allowed to call more than one friend in the same day?

(b) Later I decide that I only have time for one phone call per day. In how many ways can I make the schedule if I can only make one call per day?

(c) Since my friends Alan and Bill live in California, it is cheaper for me to call them on the weekend (Saturday and Sunday). In how many ways can I make the schedule if Alan and Bill must be called on the weekend, and I am allowed to make multiple calls per day?

(d) In how many ways can I make the schedule if Alan and Bill must be called on the weekend, and I am only allowed to make one call per day?

(e) Using your answers above, find the probability that at least two of my friends were born on the same day of the week.

8. (*20 points*) Two people, Alan and Bill share a car. Every day one of the two has the car. They agree that every night, whoever has the car has to ask the other person whether he wants the car back the following morning. Suppose that when Alan has the car and asks Bill if he wants it, he answers that he wants it 25% of the time. Also suppose that when Bill has the car and asks Alan if he wants it, he answers that he wants it 50% of the time.

(a) Draw a transition diagram for this Markov Chain.

(b) Write down the transition matrix.

(c) If Alan has the car the first day, what is the probability that Bill will have the car after two days?

(d) Is this a regular Markov Chain? Remember to justify your answer.

(e) In the long term, what is the probability that Alan will have the car in any given day?

9. (*20 points*) Three people Alan, Bill and Chuck play the following game. They take turns to toss a die. The first one who gets either a one or a six wins. Suppose that Alan starts playing, and then comes Bill, then Chuck, and then Alan again, etc...
Consider the Markov chain that has the following states:

1. It is Alan's turn to throw the die.
2. Alan has won the game
3. It is Bill's turn to throw the die.
4. Bill has won the game
5. It is Chuck's turn to throw the die.
6. Chuck has won the game

(a) Draw the transition diagram for this Markov Chain.

(b) Write down a transition matrix P in the form $\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)$.

(c) Given that $\begin{pmatrix} \frac{27}{19} & \frac{18}{19} & \frac{12}{19} \\ \frac{12}{19} & \frac{27}{19} & \frac{18}{19} \\ \frac{18}{19} & \frac{12}{19} & \frac{27}{19} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 1 & -\frac{2}{3} \\ -\frac{2}{3} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, compute F and $F\,R$.

(d) How many times is Alan expected to play before the game is over?

(e) For how long is this game expected to last?

(f) What is the probability that Alan will win the game?