Legendrian Knots

A knot is a smooth embedding \( L : S^1 \to \mathbb{R}^3 \). Consider the space \( \mathbb{R}^3 = \{(p,q,u)\} \) equipped with the standard contact form \( \alpha = du - pdq \). A smooth knot \( L \) in that space is called Legendrian if the restriction of \( \alpha \) to \( L \) vanishes.

An oriented Legendrian knot \( L \) has three classical invariants

- Its isotopy class in the space of smooth embeddings. Two Legendrian knots are said to be Legendrian isotopic if they can be connected by a path in the space of Legendrian embeddings (there exists some diffeomorphism \( g \) of \( \mathbb{R}^3 \) such that \( g \ast \alpha = \phi \alpha \), where \( \phi > 0 \)). Every smooth knot is isotopic to a Legendrian one. Two different Legendrian knots that are smooth isotopic may be not Legendrian isotopic. (It’s still an open problem to classify the Legendrian knots up to Legendrian isotopy)
- The Bennequin number \( \beta(L) \) of \( L \) is the linking number between \( L \) and \( s(L) \) with respect to the orientation defined by \( \alpha \wedge d\alpha = -dp \wedge dq \wedge du \), where \( s \) is a small shift along the \( u \)-direction.
- The Maslov number \( m(L) \) is twice the rotation number of the projection of \( L \) to the \( p - q \) plane. Think about the reason why \( m \) and \( \beta \) are invariants under Legendrian isotopy. When the orientation on \( L \) changes, the sign of \( m(L) \) changes while \( \beta(L) \) preserves.

We define two projections. \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \), \( (p,q,u) \mapsto (p,q) \) and \( \sigma : \mathbb{R}^3 \to \mathbb{R}^2 \), \( (p,q,u) \mapsto (q,u) \). We say \( L \) is generic if all self-intersections of \( \pi(L) \) are transverse double points. We draw the \( \pi \)-diagram, the projection \( \pi(L) \), in a way that at every double point the branch with larger \( u \) stay the upper one. Then
\[ \beta(L) = \#(\text{Legendrian}) - \#(\text{unknot}) \]

(Legendrian isotopic of unknots in the contact structure)

Are there Legendrian knots with the same \( m \) and \( \beta \) invariants but not Legendrian isotopic to each other?

**Theorem 1.** Legendrian knots \( L_1, L_2 \) whose \( \pi \)-diagram are given as follows have the same classical invariants, but are not Legendrian isotopic.

\[
\begin{array}{c}
\text{Legendrian isotopic of unknots in the contact structure} \\
\text{Are there Legendrian knots with the same} \\
m \text{and} \beta \text{invariants but not Legendrian isotopic} \\
to each other?
\end{array}
\]

**Differential algebra**

We construct a differential graded algebra \((A, d)\) over \( \mathbb{Z}_2 \) where \( d : A \to A \) satisfies \( d(ab) = d(a)b + ad(b) \) for \( a, b \in A \) and \( d^2 = 0 \). Denote by \( T(a_1, \ldots, a_n) \) the free associative unitary algebra over \( \mathbb{Z}_2 \) with generators \( a_1, \ldots, a_n, \ n \geq 0 \). Then \( T(a_1, \ldots, a_n) = \bigoplus_{l=0}^{\infty} A_l \) where \( A_l \) is spanned by \( l \) generators and \( A_0 \) is just \( \mathbb{Z}_2 \).

Let \( Y \subset \mathbb{R}^2 \) be the \( \pi \)-diagram of a Legendrian knot \( L \), we define the following items respectively

- \( W_k(Y) \) the collection of smooth orientation-preserving immersions \( f : \Pi_k \to \mathbb{R}^2 \) such that \( f(\partial \Pi_k \subset Y) \) where \( \Pi_k \) is a \( k \)-gon with vertices numbered counterclockwise.
- \( \tilde{W}_k(Y) = W_k(Y) / \{ g \in \text{Diff} \mid g(x^k) = x^k \} \)
- An immersion \( f \in \tilde{W}_k(Y) \) is admissible if the first vertex \( x^k_0 \) is positive and all the others are negative. The set of such \( f \)'s is denoted by \( W^+_k(Y) \).
- \( d(a_j) = \sum_{k \geq 1} \sum_{f \in W^+_k(Y, a_{j_1}, \ldots, a_{j_{k-1}})} a_{j_1} \cdots a_{j_{k-1}} \) and we extend \( d \) to a linear map \( A_Y \to A_Y \)

**Theorem 2.** \( d^2 = 0 \).

**Theorem 3.** Let \((A_{L_1}, d), (A_{L_2}, d)\) be the differential graded algebras associated with Legendrian isotopic generic Legendrian knots \( L_1, L_2 \). Then the homology rings of \((A_{L_1}, d)\) and \((A_{L_2}, d)\) are isomorphic.

**Example 4.** Let’s look at three examples.
Define the invariant $I$ of $d$ operator $d$. We have $m(L_1) = m(L_2) = 0$, $\beta(L_1) = \beta(L_2) = 1$.

- $d(a_1) = 1 + 1 = 0$
- $d(a_1) = a_4 + a_3a_4 + a_4a_5 + a_4a_5a_3a_4$
  $d(a_2) = 1 + a_3 + a_5 + a_3a_4a_5$
  $d(a_3) = d(a_4) = d(a_5) = 0$
- $d(a_1) = 1 + a_4a_6$
  $d(a_2) = 1 + a_5a_4$
  $d(a_3) = 1 + a_6a_5$
  $d(a_4) = d(a_5) = d(a_6) = 0$

We have $m(L_1) = m(L_2) = 0$, $\beta(L_1) = \beta(L_2) = 1$. For $L_1$, we have $d(a_1) = 1 + a_7 + a_7a_6a_5$, $d(a_2) = 1 + a_9 + a_5a_9a_9$, $d(a_3) = 1 + a_3a_7$, $d(a_4) = 1 + a_8a_9$, $d(a_5) = d(a_6) = d(a_7) = d(a_8) = d(a_9) = 0$; for $L_2$, we have $d(a_1) = 1 + a_7 + a_7a_6a_5 + a_5$, $d(a_2) = 1 + a_9 + a_5a_6a_9$, $d(a_3) = 1 + a_8a_7$, $d(a_4) = 1 + a_8a_9$, $d(a_5) = d(a_6) = d(a_7) = d(a_8) = d(a_9) = 0$.

Let $A = T(a_1, \ldots, a_n)$, $\bar{A} = \bigoplus_{i=1}^{\infty} A_i$. The differential algebra $(A, d)$ is called augmented if $d(\bar{A}) \subset \bar{A}$. Let $d = \sum_{i=0}^{\infty} d_i$ where $d_i(a_i) \subset A_i$ for every $i \in \{1, 2, \ldots, n\}$. Suppose $(A, d)$ is augmented, then $d_0 = 0$ and $d(\bar{A}^m) \subset \bar{A}^m = \bigoplus_{i=m}^{\infty} A_i$ for every $m$. So $d$ induces a linear operator $d^{(1)}$ on the quotient vector space $\bar{A}/\bar{A}^2$. $d^{(1)} = 0$. $d^{(1)}$ coincide with the restriction of $d_1$ to $A_1$.

Consider the cohomology of $d^{(1)}$. Let $i(A, d) = \dim(\ker d^{(1)}) - \dim(\text{im} d^{(1)}) = n - 2 \dim(\text{im} d^{(1)})$. Define the invariant $I(L) = \{i\}$ for $i(T(a_1, \ldots, a_n), d^g)$ where $d^g = gdg^{-1}$ over all $g \in \text{Aut}(A)$ such that $(A, d^g)$ is augmented.

**Theorem 5.** If $L$ is Legendrian isotopic to $L'$, then $I(L) = I(L')$. 
Compute the differential algebra \((A,d)\) for \(L_1, L_2\). Let \(g \in \text{AUT}_0(A)\) be given by \(g(a_i) = a_i + c_i, \ i \in \{1,\ldots,9\}\). \(d^\theta(a_i) = g(d(a_i))\). After computation, we have \(I(L_1) = \{3\}, I(L_2) = \{1\}\).

**Decompositions of Fronts** For a Legendrian knot \(L \subset \mathbb{R}^3\), its \(\sigma\)--projection, or front projection, \(\sigma(L) \subset \mathbb{R}^2\) is a singular curve with nowhere vertical tangent vectors. (\(q\)-axis horizontal \(u\)-axis vertical) Redefine the Maslov and Bennequin number under this projection. \(L\) is \(\sigma\)-generic if all self-intersections of \(\sigma(L)\) are transverse double points with different \(q\) coordinates. Since the overpassing branch (the one with the greater value of \(p\)) is always the one with the greater slope, so there is no need to show the type of a crossing of \(\sigma(L)\). The Maslov and Bennequin numbers can be computed as follows,

\[
m(L) = \#(\begin{array}{c} \searrow \\ \nearrow \end{array}) - \#(\begin{array}{c} \nearrow \\ \searrow \end{array}).
\]

\[
\beta(L) = \#(\begin{array}{c} \swarrow \\ \nwarrow \end{array}) + \#(\begin{array}{c} \nwarrow \\ \swarrow \end{array}) - \#(\begin{array}{c} \swarrow \\ \nwarrow \end{array}) - \#(\begin{array}{c} \nwarrow \\ \swarrow \end{array}).
\]

The four conditions of admissible decompositions

Suppose \(\Sigma = \sigma(L)\) is a union of closed curves \(X_1, \ldots, X_n\) that have finitely many intersections, then \(\{X_1, \ldots, X_n\}\) is called a decomposition of \(\Sigma\). A decomposition \(\{X_1, \ldots, X_n\}\) is called admissible if it satisfies the four conditions as follows,

- Each curve \(X_i\) bounds a topological disk \(X_i = \partial B_i\).
- For each \(i \in \{1, \ldots, n\}\), \(q \in \mathbb{R}\), the set \(B_i(q) = \{u \in \mathbb{R} \mid (q,u) \in B_i\}\) is either a segment or a single point \(u\) such that \((q,u)\) is a cusp of \(\Sigma\), or is empty. (switching/non-switching crossing point)
- If \((q_0, u) \in X_i \cap X_j (i \neq j)\) is switching then for each \(q \neq q_0\) sufficiently close to \(q_0\) the set \(B_i(q) \cap B_j(q)\) either coincide with \(B_i(q)\) or \(B_j(q)\), or is empty.
- Every switching crossing is Maslov (if \(r\) takes the same value on both its branches)

**Example 6.** Let’s look at the four examples

![Diagrams](image)

Denote by \(\text{Adm}(\Sigma)\) the set of admissible decompositions of \(\Sigma\). Given \(D \in \text{Adm}(\Sigma)\), denote by \(Sw(D)\) the set of its switching points. Define \(\theta(D) = \#(D) - \#(Sw(D))\).

**Theorem 7.** If \(\sigma\)--generic Legendrian knots \(L_1, L_2 \subset \mathbb{R}^3\) are Legendrian isotopic then there exists a one-to-one mapping \(g : \text{Adm}(\sigma(L_1)) \rightarrow \text{Adm}(\sigma(L_2))\) such that \(\theta(g(D)) = \theta(D)\) for
all $D \in \text{Adm}(\sigma(L))$. In particular, the number $\#(\text{Adm}(\sigma(L)))$ is an invariant of Legendrian isotopy.

The fronts $\Sigma_1, \Sigma_2$ correspond to the Legendrian knots $L_1, L_2$. We show that $\#(\text{Adm}(\Sigma_1)) \neq \#(\text{Adm}(\Sigma_2))$.

References