Worksheet on Chapters 1 and 2

1 Functions

Problem 1. Consider the function $f(x) = 5x + 7$. Which of the following is true

1. The graph of $f(x)$ does not intersect the line the $y = x$ and therefore it does not have an inverse function
2. The inverse function $f^{-1}(x)$ exists and therefore its graph never touches the graph of $f(x)$
3. The graph of $f^{-1}(x)$, the graph of $f(x)$ and the line $y = x$ all meet in exactly one point
4. The function $f^{-1}(x)$ exists, but only after restricting the domain of $f(x)$ appropriately

Solution: the function is invertible on the whole domain ($\mathbb{R}$) and $f^{-1}(x) = \frac{1}{5}x - \frac{7}{5}$. So the answers 2. and 4. are wrong. The graph of $f(x)$ does intersect the line $y = x$, so the answer 1. is wrong. In any case, having inverse has nothing to do with the graph intersecting the line $y = x$. So the answer is 3., which is also easily verifiable directly.

Problem 2. Find the implied domains of the following functions

1. $f(x) = \sqrt{73 - x} - \sqrt{37 + x}$
2. $f(x) = 5 \ln(x - 6)$

Solution: 1. The domain for this function is the intersection of the domains of $\sqrt{73 - x}$ and $\sqrt{37 + x}$. The domain of $\sqrt{73 - x}$ is $(-\infty, 73]$ and the domain of $\sqrt{37 + x}$ is $[-37, +\infty)$. Therefore, the answer is $[-37, 73]$. 2. The function is defined whenever the argument of $\ln$ is positive, i.e. for $x \in (6, +\infty)$.

Problem 3. Sketch the graph of $y = -\sqrt{2x - 1} + 1$ starting from the graph of $y = \sqrt{x}$.

Solution:
2 Limits

Problem 4. Multiple Choice. Consider the function

\[ f(x) = \begin{cases} 
  x^2 & \text{x rational} \\
  -x^2 & \text{x irrational} \\
  \text{undefined} & \text{x = 0}
\end{cases} \]

Then

1. There is no a for which \( \lim_{x \to a} f(x) \) exists
2. There may be an a for which \( \lim_{x \to a} f(x) \) exists, but we can’t say what it is without more information
3. \( \lim_{x \to a} f(x) \) exists for \( a = 0 \)
4. \( \lim_{x \to a} f(x) \) exists for infinitely many a

Solution: Since for any sequence \( x_n \) of inputs getting closer to 0, the sequence of outputs \( f(x_n) \) runs over either \( x_n^2 \) or \( -x_n^2 \) both of which tend to 0. So the limit \( \lim_{x \to 0} f(x) \) exists, and equals 0.

For any other point \( a \neq 0 \), if we look at a sequence of inputs \( x_n \) running over rational numbers, the outputs \( f(x_n) \) will be \( x_n^2 \) getting closer to \( a^2 \). If we look at a sequence of irrational numbers \( x_n \) getting closer to \( a \), the sequence of outputs \( f(x_n) = -x_n^2 \) will get closer to \( -a^2 \). Since \( a \neq 0 \), \( a^2 \neq -a^2 \) and so the limit does not exist.

Therefore, \( x = 0 \) is the only point where the limit exists.

Problem 5. True or False. The limit \( \lim_{x \to a} f(x) \) depends on how \( f(a) \) is defined.

Solution: False, the limit only depends on the values near \( a \), not at the point \( a \) itself.

Problem 6. True or False. If \( f(a) \) is undefined then \( \lim_{x \to a} f(x) \) cannot exist.

Solution: False, same reason as above.

Problem 7. If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \) then \( \lim_{x \to a} f(x)/g(x) \)

1. Does not exist
2. Must exist
3. Not enough information

Solution: 3. is correct. Simple examples show that the limit might exist and might not exist. If we take \( f(x) = x^2 \) and \( g(x) = x \), the limit of \( f(x)/g(x) \) exists and equals 0. If we switch the functions, take \( f(x) = x \) and \( g(x) = x^2 \), the limit will not exist.

Problem 8. \( \lim_{x \to 0} x^2 \sin(1/x) \)

1. Does not exist because no matter how close \( x \) gets to 0, there are \( x \)'s near zero for which \( \sin(1/x) \) is 1, and \( x \)'s for which \( \sin(1/x) \) is -1
2. Does not exist because the function value oscillates around 0
3. Does not exist because 1/0 is undefined
4. Equals 0
5. Equals 1

Solution: Using the Sandwich Theorem, \( -x^2 \leq x^2 \sin(1/x) \leq x^2 \) and \( \lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2) = 0 \).

Problem 9. Find the all the asymptotes of the function \( f(x) = \frac{1 + x^4}{x^2 - x^4} \).

Solution: Since \( \lim_{x \to \infty} \frac{1 + x^4}{x^2 - x^4} = -1 \), there is a horizontal asymptote \( y = -1 \). Notice that since
\[ \lim_{x \to -\infty} \frac{1 + x^4}{x^2 - x^4} = -1 \] as well, there is only one horizontal asymptote.

Vertical asymptotes correspond to points \( x = a \) with \( \lim_{x \to a^\pm} f(x) = \pm \infty \). For rational functions this can only happen at the point where the denominators becomes 0. In this case, at the points where \( x^2 - x^4 = 0 \), i.e. \( x = 0, x = \pm 1 \). It is clear that \( \lim_{x \to a} f(x) \) at each of these points is infinite. Thus we have three vertical asymptotes: \( x = 0, x = 1, x = -1 \).
3 Continuity

Problem 10. You are running a bath but you don’t close the tap properly and it is dripping. It drips once per second, each drip raising the level of the bathwater by exactly 1mm.

1. Let \( f \) be the function that represents height of the bathwater at time \( t \). Is \( f(x) \) a continuous function?
2. Let \( g \) be the function that describes the volume of water as a function of the height of the bathwater. Is \( g(x) \) a continuous function?

Solution: 1. Since the height changes in time by jumps (every time the drop falls), the function is not continuous.
2. Function \( g \) is continuous, since arbitrary small increments in height give arbitrary small increments in the volume. This function has nothing to do with the water dripping.

Problem 11. You know that

If \( f(x) \) is a polynomial function then \( f(x) \) is continuous.

Which of the following is true.

1. If \( f(x) \) is continuous then \( f(x) \) is a polynomial
2. If \( f(x) \) is not a polynomial then \( f(x) \) is not continuous
3. If \( f(x) \) is not continuous then \( f(x) \) is not a polynomial
4. All of the above

Solution: If \( f(x) \) is not continuous then \( f(x) \) is not a polynomial.

Problem 12.

1. Solve the equation \( x^2 + 13x + 41 = 1 \).
2. Use the IVT to prove that \( x^2 + 13x + 41 = \sin x \) has at least 2 solutions between the two roots found above.

Solution: 1. Solving the quadratic equation \( x^2 + 13x + 40 = 0 \) gives two solutions \( x = -8 \) and \( x = -5 \).
2. We know that the value of \( x^2 + 13x + 41 \) at \( x = -8 \) and \( x = -5 \) is 1. Therefore, the values of \( f(x) = x^2 + 13x + 41 - \sin x \) at \( x = -8 \) and \( x = -5 \) are \( 1 - \sin(-8) \) and \( 1 - \sin(-5) \) both positive since \( \sin x \) is always \( \leq 1 \).

Now, if we compute the value in the middle between \( x = -8 \) and \( x = -5 \), we get \( f(-6.5) = -1.25 - \sin(6.5) \) which is negative since \( \sin x \) is always between \(-1\) and 1. Thus, by IVT, there is a point between \( x = -8 \) and \( x = -6.5 \) where \( f(x) = 0 \), and there is a point between \( x = -6.5 \) and \( x = -5 \) where \( f(x) = 0 \).