Problems on derivatives

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Problems

Problem 1. Consider the function \( f(x) = \begin{cases} x^2, & \text{if } x \text{ rational} \\ -x^2, & \text{if } x \text{ irrational} \end{cases} \). Does the derivative \( f'(0) \) exist and why?

(a) yes;
(b) no;
(c) not possible to tell.

Solution: (a). We have seen before that the only point where \( f(x) \) is even continuous is the point \( x = 0 \). So the only point where it might potentially have derivative is \( x = 0 \). By definition,

\[
 f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h}
\]

If \( h \) is rational, \( \frac{f(h)}{h} = h \), and if \( h \) is irrational, \( \frac{f(h)}{h} = -h \). So we are really interested if the function \( g(h) = \begin{cases} h, & \text{if } h \text{ rational} \\ -h, & \text{if } h \text{ irrational} \end{cases} \) has limit when \( h = 0 \). And indeed, by the Sandwich theorem, \( g(h) \) is squeezed between \(|h|\) and \(-|h|\) so it must have limit at \( h = 0 \), which equals 0.

Problem 2. True or False: The function \( g(x) = x|x| \) does not have a derivative at \( x = 0 \), because \(|x|\) is not differentiable at \( x = 0 \).

Solution: False. By definition of derivative, \( g'(0) = \lim_{h \to 0} \frac{h|h|}{h} = \lim_{h \to 0} |h| = 0 \).

Problem 3. Find the derivative \( \frac{d}{dx}(e^x) \).

Solution: Since \( e^x \) is a constant, the derivative \( \frac{d}{dx}(e^x) = 0 \).

Problem 4. Find the equation of tangent line to the graph of \( f(x) = 2x^3 \) at the point \( x = 1 \).

Solution: The slope is \( f'(1) = 6 \cdot 1^2 = 6 \), and so the equation is \( y = 6x - 4 \).

Problem 5. Suppose you cut a slice of pizza from a circular pizza of radius \( r \), as shown.

As you change the size of the angle \( \theta \), you change the area of the slice \( A = \frac{1}{2}r^2\theta \). Then \( A' \) is

(a) \( r\theta \)
(b) \( \frac{1}{2}r^2 \).

Solution: The variable is \( \theta \) since it’s that angle that’s changing, not the radius. So the derivative is \( \frac{1}{2}r^2 \).

Problem 6. We know that for some function \( f \), \( f(1) = 1 \) and \( f'(1) = 3 \). Find the derivative of \( \frac{f(x)}{x^2} \) at \( x = 1 \).

Solution: Using the power rule, \((x^{-2}f(x))' = -2x^{-3}f(x) + x^{-2}f'(x)\), and so the derivative of \( \frac{f(x)}{x^2} \) at \( x = 1 \) is \( -2 \cdot 1 \cdot f(1) + 1 \cdot f'(1) = -2 + 3 = 1 \).
Problem 7. Differentiate $f(x) = \frac{x^3}{3x-1}$.

Solution: Using the division rule, we get $f'(x) = \frac{2x(3x-1)-3x^3}{(3x-1)^2} = \frac{-2x}{(3x-1)^2}$.

Problem 8. Calculate $f^{(n)}(0)$ for $f(x) = x^n e^x$.

Solution: Do it for small $n$ first, for example for $n = 1, 2, 3$. In general, we notice that the only time a term will not die after plugging in $x = 0$ is when it’s of the form const $\cdot e^x$. The only such term is $n!e^x$, and so $f^{(n)}(0) = n!$. 