Some problems from Prelim 2 of Fall 2013

October 31, 2016

Problem 1. (d) We need to find derivative of $y = x^2$. Using, for example, logarithmic differentiation we get $\ln y = 2x \ln x$, and so

\[
\frac{(\ln y)'}{y} = \frac{1}{y} \cdot y' = 2x \ln 2 \ln x + 2x \cdot \frac{1}{x} = 2x (\ln 2 \ln x + \frac{1}{x})
\]

So, we have $\frac{1}{y} \cdot y' = 2x (\ln 2 \ln x + \frac{1}{x})$, and therefore $y' = x^2 \cdot 2x (\ln 2 \ln x + \frac{1}{x})$.

Problem 2. (a) The formula for the area of a circle is $A = \pi r^2$. We need to find the rate of change $\frac{dA}{dt}$ after 2 hours. We have $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$. To find $\frac{dr}{dt}$ we can relate $r$ to $h$, which will help since we are given that $\frac{dh}{dt} = 0.5$. Using similar triangles (see picture below) we get $\frac{3}{10} = \frac{r}{h}$, i.e. $r = \frac{3}{10} h$.

Therefore, $\frac{dr}{dt} = \frac{3}{10} \frac{dh}{dt} = \frac{3}{20}$. Finally, after 2 hours the height will be $h = 2 \cdot 0.5 = 1m$, and so $r$ will be $r = \frac{3}{10} \cdot 1 = \frac{3}{10}$. Putting it all together, we get

\[
\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt} = 2\pi \cdot \frac{3}{10} \cdot \frac{3}{20} = \frac{9\pi}{100}
\]

(b) Now we are interested in computing the rate of change of the volume $V = \frac{1}{3}\pi r^2 h$ at the moment when the hole fills up, i.e. at the moment when $h = 10$ and $r = 3$. Differentiating and using the information we have found in part (a), we get

\[
\frac{dV}{dt} = \frac{2}{3} \pi r h \frac{dr}{dt} + \frac{1}{3} r^2 \frac{dh}{dt} = \frac{2}{3} \pi \cdot 3 \cdot 10 \cdot \frac{3}{20} + \frac{1}{3} \pi \cdot 3^2 \cdot \frac{1}{2} = \frac{9\pi}{2}
\]
Problem 4. (c) We need to find the derivative of \( p(x) = \sqrt{f^{-1}(x)} + 1 \) at \( x = 2 \) using the information in the following table:

<table>
<thead>
<tr>
<th></th>
<th>f(x)</th>
<th>g(x)</th>
<th>f’(x)</th>
<th>g’(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>−6</td>
<td>1/2</td>
<td>10</td>
<td>−7</td>
</tr>
<tr>
<td>−1</td>
<td>−2</td>
<td>0</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>−1</td>
<td>1/3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5/2</td>
<td>−4</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Using the chain rule,

\[
\frac{dp}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{f^{-1}(x)} + 1} \cdot \frac{d}{dx} (f^{-1}(x) + 1) = \frac{1}{2} \cdot \frac{1}{\sqrt{f^{-1}(x)} + 1} \cdot \frac{1}{f'(f^{-1}(x))}.
\]

Therefore, at \( x = 2 \) we will have

\[
\frac{dp}{dx}(2) = \frac{1}{2} \cdot \frac{1}{\sqrt{0} + 1} \cdot \frac{1}{f'(f^{-1}(2))}.
\]

Since we are given \( f(0) = 2 \) we know that \( f^{-1}(2) = 0 \). As a result,

\[
\frac{dp}{dx}(2) = \frac{1}{2} \cdot \frac{1}{\sqrt{0} + 1} \cdot \frac{1}{f'(0)} = \frac{3}{2}.
\]