Problems

Problem 1. Compute the integral \( \int_{1}^{2} x^k \, dx \) by definition using the following partition.

For \( r := 2^{1/n} \), define the partition to be \( 1 < r < r^2 < \cdots < r^{n-1} < r^n = 2 \). The purpose of this problem is to show that a particular choice of the partition to work with can make a huge difference. Try to compute this integral by using the standard partition and see for yourself how complicated it will be.

Problem 2. True or False:

1. If \( \int f(x) \, dx = \int g(x) \, dx \) for continuous functions \( f \) and \( g \), then \( f(x) = g(x) \).

2. If \( f \) and \( g \) are differentiable and \( f'(x) = g'(x) \), then \( f(x) = g(x) \).

3. \( \frac{d}{dx} \left( \int_{0}^{x} f(t) \, dt \right) = f(x) \).

Problem 3. Below is the graph of a function \( f \).

\[
\begin{array}{c}
\text{Let } g(x) = \int_{0}^{x} f(t) \, dt \text{ then for } 0 < x < 2 \text{ the function } g(x) \text{ is}
\end{array}
\]

1. increasing and concave up;

2. increasing and concave down;

3. decreasing and concave up;

4. decreasing and concave down.

Problem 4. Evaluate the area bounded by the curve \( x = 2 - y - y^2 \) and the \( y \)-axis. (Hint: try switching the roles of the coordinates.)
Problem 5. Compute the area bounded by the curve $y = \frac{1}{x}$, the x-axis, the straight line $x = e$ and the horizontal line $y = 2$. (Hint: draw the picture.)

Problem 6. Using definite integrals, find the limit of the following sum:

$$
\lim_{n \to \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right)
$$

(Hint: $\frac{1}{n+i} = \frac{1}{n} \cdot \frac{1}{1 + \frac{i}{n}}$)

Problem 7. Determine the signs of the following integrals without evaluating them:

1. $\int_{-1}^{3} x^2 \, dx$
2. $\int_{0}^{2\pi} \frac{\sin x}{x} \, dx$

Problem 8. Determine (without evaluating) which of the following integrals is greater:

1. $\int_{0}^{1} \sqrt{1 + x^2} \, dx$ or $\int_{0}^{1} \, dx$
2. $\int_{0}^{1} x^2 \sin^2 x \, dx$ or $\int_{0}^{1} x \sin^2 x \, dx$

Problem 9. Let $f$ be a function on $[0, 1]$ defined as follows:

$$
f(x) = \begin{cases} 
\frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ is a rational number, } \text{g. c. d.}(p, q) = 1 \\
0, & \text{if } x \text{ is irrational}
\end{cases}
$$

If this function integrable? (Hint: try to pick specific partitions similar to the example of Dirichlet function.)