Problems

Problem 1. Two horses start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. (Hint: MVT.)

Solution: Let \( s_1(t) \) and \( s_2(t) \) be two functions that represent the position of the first and the second horse from the start at time \( t \), respectively. Since both horses start at the same time, \( s_1(0) = s_2(0) = 0 \) (since they are both at the start at \( t = 0 \)). Since they finish at a tie, this means it took them the same time \( t_0 \) to get to the finish, and so \( s_1(t_0) = s_2(t_0) \). Consider the function \( d(t) = s_1(t) - s_2(t) \). By MVT (assuming differentiability of both \( s_1(t) \) and \( s_2(t) \), of course) we get

\[
\frac{d(t_0) - d(0)}{t_0 - 0} = d'(c) = s_1'(c) - s_2'(c)
\]

for some \( c \) between 0 and \( t_0 \). But \( d(t_0) = s_1(t_0) - s_2(t_0) = 0 \) and \( d(0) = s_1(0) - s_2(0) = 0 \), so at \( c \) we have \( s_1'(c) = s_2'(c) \). This is exactly what we wanted, since the derivative of the distance is the velocity.

Problem 2. Let \( f(x) = \frac{1}{x^2} \), and \( F(x) \) be an antiderivative of \( f \) with the property \( F(1) = 1 \). True or false: \( F(-1) = 3 \).

Solution: False. Since \( f(x) \) is not continuous on \( \mathbb{R} \), we can pick different constants on \((-\infty, 0)\) and \((0, +\infty)\). In other words, an anti-derivative of \( f(x) \) does not have to be of the form \( F(x) = -\frac{1}{x} + C \). If it were, then \( F(1) = -1 + C = 1 \) so \( C = 2 \) would give \( F(-1) = -\frac{1}{-1} + 2 = 3 \). However, one can take, for example,

\[
F(x) = \begin{cases} 
-\frac{1}{x} + 2 & \text{if } x > 0 \\
-\frac{1}{x} + 100 & \text{if } x < 0 
\end{cases}
\]

This is a perfectly good anti-derivative of \( f(x) \) with \( F(1) = 1 \) but \( F(-1) \neq 3 \).

Problem 3. A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec\(^2\). How fast will the rocket be going 1 minute later?

Solution: Acceleration is given by the derivative of the velocity, \( a(t) = \frac{dv(t)}{dt} \). We are given \( a(t) = \frac{dv(t)}{dt} = 20 \) and so \( v(t) = 20t + C \) for some constant \( C \). Since at the time \( t = 0 \) the rocket is not moving, \( v(0) = 0 \), i.e. \( C = 0 \). This gives \( v(t) = 20t \). In one minute, the speed will be \( v(60) = 1200 \) m/sec\(^2\).

Problem 4. Compute the sum \( \sum_{i=3}^{n} (i - 2)^2 \). You can use the formulas we’ve seen in class.

Solution: \( \sum_{i=3}^{n} (i - 2)^2 = 1^2 + 2^2 + \cdots + (n - 2)^2 = \sum_{i=1}^{n-2} i^2 = \frac{(n-2)(n-1)(2n-3)}{6} \).
Problem 5. Compute \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} \). (Hint: \( \frac{1}{2} - \frac{1}{3} = \frac{1}{2} - \frac{1}{3} \))

Solution:

\[
\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}
\]

The last equality is just the result of cancellations: the whole sum “telescopes.” Finally,

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1
\]

Problem 6. Compute the integral \( \int_{0}^{2} x \, dx \) by definition. Verify that this answer is the same as the usual (geometric) area under the graph of \( f(x) = x \) over \([0, 2]\).

Solution: Because \( f(x) = x \) is continuous, it is also integrable, and so it does not matter which Riemann sums to consider. For example, take the upper sum. Split \([0, 2]\) into \( n \) intervals of equal length \( \frac{2}{n} \). Then \( i \)-th interval will be \([\frac{2(i-1)}{n}, \frac{2i}{n}]\). On this interval, the maximum of the function \( f(x) = x \) is attained at the rightmost point \( \frac{2i}{n} \), and this maximum value is \( \frac{2i}{n} \). Thus, the upper sum is

\[
U_n = \sum_{i=1}^{n} \frac{2i}{n} \cdot \frac{2}{n} = \frac{4}{n^2} \sum_{i=1}^{n} i = \frac{4}{n^2} \cdot \frac{n(n+1)}{2} = \frac{2(n+1)}{n}
\]

Thus, \( \lim_{n \to \infty} U_n = \lim_{n \to \infty} \frac{2(n+1)}{n} = 2 \). This coincides with the usual area of the triangle.

Problem 7. We cut a circular disk of radius \( r \) into \( n \) circular sectors, as shown in the figure, by marking the angles \( \theta_i \) at which we make the cuts (\( \theta_0 = \theta_n \) can be considered to be angle 0). A circular sector between two angles \( \theta_i \) and \( \theta_{i+1} \) has area \( \frac{1}{2} r^2 \Delta \theta \), where \( \Delta \theta = \theta_{i+1} - \theta_i \).

We let \( A_n = \sum_{i=0}^{n-1} \frac{1}{2} r^2 \Delta \theta_i \). Then the area of the disk, \( A \), is given by

1. \( A_n \), independent of how many sectors we cut the disk into.
2. \( \lim_{n \to \infty} A_n \).
3. \( \int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta \).
4. all of the above.

Solution: all of the above. Clearly 2. and 3. are equivalent. But \( \lim_{n \to \infty} A_n = A_n \) since both equal simply to the area of the disk: the sum of the areas of sectors is the area of the whole disk.