Let $G$ be a group, and $H$ be a subgroup. The subgroup $H$ is called **normal** if for any $g \in G$ we have $gHg^{-1} = H$ (equality of sets!). In other words, $H$ is normal if and only if all left cosets are the same as right cosets, $gH = Hg$.

**Problem 1.** For a group $G$ prove the following.

1. If $G$ is abelian, every its subgroup is normal.
2. The two subgroups $\{e\}$ and $G$ itself are both normal.
3. The **center** $Z(G) = \{g \in G \mid \forall h \in G, hg = gh\}$ is a normal subgroup.

**Problem 2.** The group $D_{2n}$ of symmetries of a regular $n$-gon has normal subgroup $\mathbb{Z}/n$ consisting of rotations.

**Problem 3.** The group $A_n$ of even permutations is normal in $S_n$.

**Problem 4.** For any (finite) group $G$ and a subgroup $H$ of index 2, $H$ is necessarily normal in $G$.

The point is: if $H$ is normal, the set of cosets $G/H$ has a natural group structure. This group is called the **quotient group**. We define $aH \ast bH := abH$. Why is it well-defined?

The next three exercises show that normal subgroups are essentially the same as kernels of homomorphisms. In other words, for a group $G$ there is a bijection between the set of normal subgroups $H \subset G$ and **surjective** homomorphisms $G \to K$.

**Problem 5.** For any homomorphism $\varphi : G \to K$, $\ker \varphi$ is a normal subgroup.

**Problem 6.** If $\varphi : G \to K$ is a surjective homomorphism, then $K \simeq G/\ker \varphi$.

**Problem 7.** The **First Isomorphism Theorem**. For any homomorphism $\varphi : G \to K$, there is an isomorphism $G/\ker \varphi \simeq \text{im } \varphi$.

Therefore, the image of any homomorphism really “looks like” a quotient group. (Compare with a similar result about the orbits of group actions we have seen before.)

**Problem 8.** Let $n\mathbb{Z} \subset \mathbb{Z}$ be the subgroup $\{\ldots, -n, 0, n, 2n, \ldots\}$. Prove that $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n$ (Hint: use the First Isomorphism Theorem).

**Problem 9.** Prove that $S_n/A_n \simeq \mathbb{Z}/2$.

**Problem 10.** Prove that $G \times H/H \simeq G$, where $H \subset G \times H$ is the subgroup $H = \{(e, h) \mid h \in H\}$.

Let $G$ be a group and $K$ be another group, on which $G$ acts by **automorphisms**. In other words, for each $g \in G$ we have assigned an isomorphism $A_g : K \to K$, such that $A_e = \text{id}$ and $A_g \circ A_h = A_{gh}$. We write $g \cdot k$ (or $g.k$) for $A_g(k)$.

We define **semi-direct product** $G \ltimes K$ to be the set $K \times G$ with the operation

$$(k_1, g_1) * (k_2, g_2) = (k_1 g_1 k_2, g_1 \cdot g_2)$$

**Problem 11.** Prove that $G \ltimes K$ is again a group, with $K = \{(k, e) \mid k \in K\}$ being a normal subgroup in $G \ltimes K$, and $G \ltimes K/K \simeq G$.

**Problem 12.** If $G$ acts trivially on $K$, then $G \ltimes K \simeq G \times K$.

**Problem 13.** The group $D_{2n}$ is isomorphic to $\mathbb{Z}/2 \ltimes \mathbb{Z}/n$ with the action of $1 \in \mathbb{Z}/2$ given by $a \mapsto -a$.

**Problem 14.** The group $S_n$ is $\mathbb{Z}/2 \ltimes A_n$, where the action is by conjugation by any odd permutation.

**Problem 15.** The group of symmetries of a cube is isomorphic to the semi-direct product $S_3 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$. (Hint: think of vertices of a cube as sequences $(\pm 1, \pm 1, \pm 1)$.)

Try to see what these groups mean geometrically. Therefore, you get an isomorphism of groups

$$S_4 \times \mathbb{Z}/2 \simeq S_3 \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2)$$