Problem 1. Let $G$ be a group. Show that if $G$ is non-abelian, then the maps defined by $g.a = ag$ for all $g, a \in G$ do not satisfy the axioms of a (left) group action of $G$ on itself.

Problem 2. Let $S_n$ act on $\mathbb{R}^n$ by permuting coordinates of vectors: for $\sigma \in S_n$ and $v = (c_1, \ldots, c_n)$ let $\sigma.v$ to be $\sigma.v = (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$. Check that this is not an action of $S_n$ on $\mathbb{R}^n$ according to our definition! (Hint: take $n = 3$, $\sigma_1 = (12)$ and $\sigma_2 = (23)$ and see that something goes wrong.)

What goes wrong in general?

Prove that if we put $\sigma.v$ to be $\sigma.v = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})$, then we will actually get an action as defined in the lectures.

Problem 3. Prove that any group $G$ acts on itself via $g.x := gxg^{-1}$.

The action of $G$ on itself via $g.x := gxg^{-1}$ is called conjugation. Orbits of this action are called conjugacy classes.

Problem 4. Let $G$ be any abelian group. Describe its conjugacy classes.

Prove that two permutations $s, t \in S_n$ belong to the same conjugacy class if and only if they have the same number of cycles of each length. Thus, the number of conjugacy classes in $S_n$ is the number $p(n)$ of partitions of $n$.

Find conjugacy classes in $S_3$ and $S_4$.

Problem 5. Find conjugacy classes in the alternating group $A_5$. Are cycles $(12345)$ and $(12354)$ conjugate in $A_5$?

Problem 6. Let $GL_2(\mathbb{R})$ act on $\mathbb{R}^2$ in the usual way, i.e.

\[
\begin{bmatrix}
    a & b \\
    c & d
\end{bmatrix} \cdot \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} = \begin{bmatrix}
    ax_1 + bx_2 \\
    cx_1 + dx_2
\end{bmatrix}
\]

Describe all orbits and stabilizers of this action. (Hint: there are two of them.)

Problem 7. Let $GL_2(\mathbb{Z})$ be the group of invertible $2 \times 2$ matrices with integer coefficients whose inverse also has integral coefficients. Let $GL_2(\mathbb{Z})$ act on $\mathbb{Z}^2$ the same way as $GL_2(\mathbb{R})$ acts on $\mathbb{R}^2$. Describe all orbits of this action and all stabilizers. (Hint: an orbit consists of vectors, whose coordinates have the same g.c.d.)