Isomorphisms

Sasha Patotski
Cornell University
ap744@cornell.edu

November 23, 2015
Last time

Definition

If $G, H$ are groups, a map $\varphi : G \to H$ is called a **homomorphism** if

$$\varphi(a \ast b) = \varphi(a) \ast \varphi(b)$$

For a homomorphism $\varphi$, necessarily $\varphi(e_G) = e_H$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Definition

The **kernel** of a homomorphism $\varphi$ is $\ker \varphi = \{ a \in G \mid \varphi(a) = e \}$

The **image** of a homomorphism $\varphi$ is $\text{im} \varphi = \{ \varphi(a) \in H \mid a \in G \}$

$\ker \varphi$ and $\text{im} \varphi$ are subgroups of $G$ and $H$, respectively.
Examples

- Between \textbf{any} groups \(G, H\) there is a \textbf{trivial} homomorphism \(\varphi : G \rightarrow H\), given by \(\varphi(g) = e_H\), for all \(g \in G\).
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- The map $n \mapsto n \pmod{m}$ defines a homomorphism $\mathbb{Z} \to \mathbb{Z}/m$.
- There are no nontrivial homomorphisms $\mathbb{Z}/m \to \mathbb{Z}$.
- For a fixed $m \in \mathbb{Z}$, the map $\varphi_m : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi_m(n) = nm$ defines a homomorphism.
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- For any abelian group $G$, the map $\varphi_m : G \to G$ given by $g \mapsto g^m$ is a homomorphism.
Between any groups $G, H$ there is a trivial homomorphism $\varphi: G \to H$, given by $\varphi(g) = e_H$, for all $g \in G$.

The map $n \mapsto n \pmod{m}$ defines a homomorphism $\mathbb{Z} \to \mathbb{Z}/m$.

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The same map for non-abelian group is not necessarily a homomorphism (can you give an example?).
Examples

- The group of symmetries of an equilateral triangle is isomorphic to $S_3$. 

Prove that the alternating group $A_3$ is isomorphic to $\mathbb{Z}/3$. 

Prove that orientation preserving symmetries of the square form a subgroup of the group of all symmetries. 

Prove that this subgroup is isomorphic to $\mathbb{Z}/4$. 

Describe the group of orientation preserving symmetries of a regular $n$-gon.
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- Describe the group of orientation preserving symmetries of a regular $n$-gon.
The group $G$ of symmetries of a regular tetrahedron is isomorphic to $S_4$. 

There is an obvious homomorphism $\phi : G \to S_4$, sending a symmetry to corresponding permutation of vertices. It is injective: if a symmetry fixes all vertices, it must be the identity symmetry. Every transposition of neighbors is in the image $\text{im}(\phi)$. But $\text{im}(\phi)$ is a subgroup. Since $S_4$ is generated by these transpositions, $\text{im}(\phi) = S_4$. Done.
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Theorem

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