Fifteen puzzle.

Sasha Patotski
Cornell University
ap744@cornell.edu

November 16, 2015
Last time

Definition

The permutation group $S_n$ is the group of bijections of the set \{1, 2, \ldots, n\}.

It is convenient to denote permutations by

$$
\sigma = \begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
\sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(n)
\end{pmatrix}
$$

Definition

For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the **number of inversions** of $\sigma$.

Definition

Define the **sign** of $\sigma$ to be $\text{sgn}(\sigma) = (-1)^{\text{inv}(\sigma)}$. 

Sasha Patotski  (Cornell University)  
Fifteen puzzle.  
November 16, 2015 2 / 7
Sign of a permutation

**Definition**

For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the **number of inversions** of $\sigma$. 

What is the sign of $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 4\ 3\ 1\ 6\ 7\ 2\ 5)$? 

Prove that for any permutation $\sigma$, composing it with a transposition of neighbors $(i, i+1)$ either creates a new inversion, or removes one. Thus composing any permutation $\sigma$ with $(i, i+1)$ changes its sign, i.e. $\text{sgn}((i, i+1) \circ \sigma) = -\text{sgn}(\sigma)$. 

Thus for any representation of $\sigma$ as a composition of $N$ transpositions of neighbors, the sign $\text{sgn}(\sigma)$ is $(-1)^N$. (Need to be careful here.) 

Prove that for two permutations $\sigma, \tau$ we have $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma) \text{sgn}(\tau)$. 

Sasha Patotski (Cornell University)
Sign of a permutation

**Definition**
For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the **number of inversions** of $\sigma$.

- What is the sign of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix}$?
Sign of a permutation

Definition
For \( \sigma \in S_n \) define \( \text{inv}(\sigma) \) to be the number of pairs \((ij)\) such that \( i < j \) but \( \sigma(i) > \sigma(j) \). This number \( \text{inv}(\sigma) \) is called the number of inversions of \( \sigma \).

- What is the sign of \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix} \)?
- Prove that for any permutation \( \sigma \), composing it with a transposition of neighbors \((i, i+1)\) either creates a new inversion, or removes one.
Sign of a permutation

Definition

For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the number of inversions of $\sigma$.

- What is the sign of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix}$?

- Prove that for any permutation $\sigma$, composing it with a transposition of neighbors $(i, i+1)$ either creates a new inversion, or removes one.

- Thus composing any permutation $\sigma$ with $(i, i + 1)$ changes its sign, i.e. $\text{sgn}((i, i+1) \circ \sigma) = -\text{sgn}(\sigma)$. 

Sasha Patotski (Cornell University)
Sign of a permutation

Definition

For $\sigma \in S_n$ define $inv(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $inv(\sigma)$ is called the number of inversions of $\sigma$.

- What is the sign of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix}$?
- Prove that for any permutation $\sigma$, composing it with a transposition of neighbors $(i, i+1)$ either creates a new inversion, or removes one.
- Thus composing any permutation $\sigma$ with $(i, i+1)$ changes its sign, i.e. $sgn((i, i+1) \circ \sigma) = -sgn(\sigma)$.
- Thus for any representation of $\sigma$ as a composition of $N$ transpositions of neighbors, the sign $sgn(\sigma)$ is $(-1)^N$. (Need to be careful here.)
Sign of a permutation

Definition

For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the number of inversions of $\sigma$.

- What is the sign of $\sigma = \left(\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
4 & 3 & 1 & 6 & 7 & 2 \\
\end{array}\right)$?

- Prove that for any permutation $\sigma$, composing it with a transposition of neighbors $(i, i + 1)$ either creates a new inversion, or removes one.

- Thus composing any permutation $\sigma$ with $(i, i + 1)$ changes it’s sign, i.e. $\text{sgn}((i, i + 1) \circ \sigma) = -\text{sgn}(\sigma)$.

- Thus for any representation of $\sigma$ as a composition of $N$ transpositions of neighbors, the sign $\text{sgn}(\sigma)$ is $(-1)^N$. (Need to be careful here.)

- Prove that for two permutations $\sigma, \tau$ we have $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma)\text{sgn}(\tau)$. 

Sasha Patotski  (Cornell University)  Fifteen puzzle.  November 16, 2015
Definition

If $sgn(\sigma) = 1$, $\sigma$ is called an even permutation, otherwise it’s called odd.

Prove that any transposition is an odd permutation.

Prove that any cycle of an even length is an odd permutation, and vice versa. (Hint: decompose a cycle as a composition of transpositions.)

Thus $sgn(\sigma) = (-1)^r$ where $r$ is the number of cycles of even lengths in the cycle decomposition of $\sigma$.

Check that it works for $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)\ (4\ 3\ 1\ 6\ 7\ 2\ 5)$.
Sign of a permutation

**Definition**

If $sgn(\sigma) = 1$, $\sigma$ is called an **even permutation**, otherwise it’s called **odd**.

- Prove that any transposition is an odd permutation.
Definition

If $\text{sgn}(\sigma) = 1$, $\sigma$ is called an **even permutation**, otherwise it’s called **odd**.

- Prove that any transposition is an odd permutation.
- Prove that any cycle of an even length is an odd permutation, and vice versa.
  (Hint: decompose a cycle as a composition of transpositions.)
Sign of a permutation

Definition
If \( sgn(\sigma) = 1 \), \( \sigma \) is called an **even permutation**, otherwise it’s called **odd**.

- Prove that any transposition is an odd permutation.
- Prove that any cycle of an even length is an odd permutation, and vice versa.
  
  (Hint: decompose a cycle as a composition of transpositions.)
- Thus \( sgn(\sigma) = (-1)^r \) where \( r \) is the number of cycles of even lengths in the cycle decomposition of \( \sigma \).
Sign of a permutation

**Definition**

If \( sgn(\sigma) = 1 \), \( \sigma \) is called an **even permutation**, otherwise it’s called **odd**.

- Prove that any transposition is an odd permutation.
- Prove that any cycle of an even length is an odd permutation, and vice versa.
  (Hint: decompose a cycle as a composition of transpositions.)
- Thus \( sgn(\sigma) = (-1)^r \) where \( r \) is the number of cycles of even lengths in the cycle decomposition of \( \sigma \).
- Check that it works for \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix} \)
Sign of a permutation

**Definition**

If \( sgn(\sigma) = 1 \), \( \sigma \) is called an **even permutation**, otherwise it’s called **odd**.

- Prove that any transposition is an odd permutation.
- Prove that any cycle of an even length is an odd permutation, and vice versa.
  (Hint: decompose a cycle as a composition of transpositions.)
- Thus \( sgn(\sigma) = (-1)^r \) where \( r \) is the number of cycles of even lengths in the cycle decomposition of \( \sigma \).
- Check that it works for \( \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 6 & 7 & 2 & 5 \end{pmatrix} \)

**Definition**

Let \( A_n \subset S_n \) be the subset consisting of even permutations. \( A_n \) is called an **alternating GROUP** (check that it’s a group!)
The Fifteen puzzle
Sam Loyd’s puzzle

A $1,000.00 Cash Prize Puzzle

Sam Loyd’s Famous 14-15 Puzzle
— Can You Do It and Win $1,000.00

FIG. 1: SHOWING ORIGINAL POSITIONS OF THE BLOCKS IN EACH PROBLEM
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$. 

Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile. Consider the number $X = s + r \mod 2$.

Compute all these numbers for the position on the pictures before. What happens to $X$ if you move the empty tile horizontally? Vertically?

Prove that Sam Loyd’s puzzle can’t be solved.
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.
- Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.

Consider the number $X = s + r \mod 2$.

Compute all these numbers for the position on the pictures before.

What happens to $X$ if you move the empty tile horizontally? Vertically?

Prove that Sam Loyd’s puzzle can’t be solved.
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.
- Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.
- Consider the number $X = s + r \mod 2$. 

Compute all these numbers for the position on the pictures before.

What happens to $X$ if you move the empty tile horizontally?

Vertically?

Prove that Sam Loyd’s puzzle can’t be solved.
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.
- Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.
- Consider the number $X = s + r \mod 2$.
- Compute all these numbers for the position on the pictures before.

What happens to $X$ if you move the empty tile horizontally? Vertically?

Prove that Sam Loyd's puzzle can't be solved.
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.
- Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.
- Consider the number $X = s + r \mod 2$.
- Compute all these numbers for the position on the pictures before.
- What happens to $X$ if you move the empty tile horizontally?
Solution to the Sam Loyd’s puzzle

- Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.
- Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.
- Consider the number $X = s + r \mod 2$.
- Compute all these numbers for the position on the pictures before.
- What happens to $X$ if you move the empty tile horizontally?
- Vertically?
Reading the puzzle left to right, top to bottom gives an element of $S_{15}$.

Let $s$ be the sign of the permutation you get, and let $r$ be the number of the row containing the empty tile.

Consider the number $X = s + r \mod 2$.

Compute all these numbers for the position on the pictures before.

What happens to $X$ if you move the empty tile horizontally?

Vertically?

Prove that Sam Loyd’s puzzle can’t be solved.