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When the bell is nearly upside down, a little tug on the rope adds enough momentum to ensure that, in a few seconds, it swings all the way in the other direction so that its near upside down again. From then on, the bell can be kept ringing with comparatively small tugs on the rope at the appropriate times.
The fact that only small adjustments can be made in the timing of when each bell rings means that the position of a bell in the extent can only be changed by one place at a time.

In other words, the physical constraints of bell-ringing imply that the only way we can change the order of bells is by swapping the positions of two bells that are one after the other.

Summing up:

(Optional:) want to ring all possible permutations of $n$ bells, starting with the trivial permutation.

Don't want any repetitions.

From one change to the next, any bell can move by at most one position in its order of ringing.

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Example: Plain Bob
Some facts about bell ringing

- To ring one change takes roughly two seconds, about the time for a large bell to complete its natural swing.
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- This means that a serious bell ringer must effectively recite a sequence of several thousand numbers, one every two seconds, and to translate this sequence into perfect bell ringing.
- If we have a large number of bells, it is not obvious how to construct an extent that obeys the bell ringing rules.
### One way to construct a sequence

<table>
<thead>
<tr>
<th>3 bells</th>
<th>4 bells</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>1234</td>
</tr>
<tr>
<td>123</td>
<td>1243</td>
</tr>
<tr>
<td>213</td>
<td>2314</td>
</tr>
<tr>
<td>231</td>
<td>2431</td>
</tr>
<tr>
<td>321</td>
<td>4123</td>
</tr>
<tr>
<td>312</td>
<td>4231</td>
</tr>
<tr>
<td>132</td>
<td>4213</td>
</tr>
<tr>
<td>(123)</td>
<td>2413</td>
</tr>
<tr>
<td></td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>2134</td>
</tr>
<tr>
<td></td>
<td>(1234)</td>
</tr>
</tbody>
</table>
More fun facts

<table>
<thead>
<tr>
<th>bells</th>
<th>name</th>
<th>changes in extent</th>
<th>time required</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Singles</td>
<td>6</td>
<td>12 seconds</td>
</tr>
<tr>
<td>4</td>
<td>Minimus</td>
<td>24</td>
<td>48 seconds</td>
</tr>
<tr>
<td>5</td>
<td>Doubles</td>
<td>120</td>
<td>4 minutes</td>
</tr>
<tr>
<td>6</td>
<td>Minor</td>
<td>720</td>
<td>24 minutes</td>
</tr>
<tr>
<td>7</td>
<td>Triples</td>
<td>5 040</td>
<td>2 hours 48 minutes</td>
</tr>
<tr>
<td>8</td>
<td>Major</td>
<td>40 320</td>
<td>22 hours 24 minutes</td>
</tr>
<tr>
<td>9</td>
<td>Caters</td>
<td>362 880</td>
<td>8 days 10 hours</td>
</tr>
<tr>
<td>10</td>
<td>Royal</td>
<td>3 628 800</td>
<td>84 days</td>
</tr>
<tr>
<td>11</td>
<td>Cinques</td>
<td>39 916 800</td>
<td>2 years 194 days</td>
</tr>
<tr>
<td>12</td>
<td>Maximus</td>
<td>479 001 600</td>
<td>30 years 138 days</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>20 922 789 888 000</td>
<td>1 326 914 years</td>
</tr>
</tbody>
</table>

An extent on eight bells was rang only once, at the Loughborough Bell Foundry in 1963. The ringing began at 6.52am on July 27, and finished at 12.50am on July 28, after 40,320 changes and 17 hours 58 minutes of continuous ringing.
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Look at the separation of the Plain Bob extent into 3 columns. They are cosets of $D_4$ as a subgroup of $S_4$. 

\begin{align*}
&1234 & 2143 & 2413 & 4231 & 4321 & 3412 & 3142 & 1324 \\
&3 & 4 & 1 & 2 & 4 & 3 & 1 & 4
\end{align*}
Think of permutations as **vertices of a graph**.

Two vertices are connected by an edge if there is a permitted transition (according to bell ringers) that transforms one change into the other. Here what it looks like for Plain Bob:
An extent is simply a path in this graph, visiting each of the vertices exactly once, and returning to the beginning vertex. Such tours are called Hamiltonian cycles.

For Plain Bob, this path looks like that:
Let $G$ be a group, and let $S$ be a generating set of elements.

**Definition**

Let $\text{Cay}(G, S)$ be the colored directed graph having $G$ as the set of vertices, and for any $s \in S$ there is an edge going from $g$ to $gs$, and any such edge is colored into a unique color $c_s$ corresponding to $s \in S$. 
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Theorem. The graph $\text{Cay}\left(\{(1, 0), (0, 1)\} : \mathbb{Z}/m \times \mathbb{Z}/n\right)$ does not have a Hamiltonian circuit when $m$ and $n$ are relatively prime and greater than 1.

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Sasha Patotski  (Cornell University)
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