Weighted Polya Theorem. Solitaire

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Cosets

- For a group $G$ and a subgroup $H \subset G$, cosets are subsets of $G$ of the form $gH$ and $Hg$ for $g \in G$.
- Let $G$ act on a set $X$, pick a point $x \in X$ and let $Gx$ and $G_x$ be its orbit and stabilizer.

**Lemma 1.** The orbit $Gx$ is in a natural bijection with the set of cosets $G/G_x = \{gG_x \mid g \in G\}$. In particular, for finite groups, $|Gx| = |G|/|G_x|$.

**Lemma 2.** For any other point $y \in Gx$ of the orbit of $x$, the stabilizer of $G_y$ is $G_y = gG_xg^{-1}$ for some $g \in G$. In particular, for finite groups, all the stabilizers of points from the same orbit have the same number of elements.
Theorem

Suppose that a finite group \( G \) acts on a finite set \( X \). Then the number of colorings of \( X \) in \( n \) colors inequivalent under the action of \( G \) is

\[
N(n) = \frac{1}{|G|} \sum_{g \in G} n^{c(g)}
\]

where \( c(g) \) is the number of cycles of \( g \) as a permutation of \( X \).
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- Let’s instead count the pairs $(g, C)$ with $C \in X_n$ a coloring and $g \in G_C \subset G$ an element of $G$ preserving $C$. 
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- Let’s instead count the pairs $(g, C)$ with $C \in X_n$ a coloring and $g \in G_C \subset G$ an element of $G$ preserving $C$.
- The orbit $GC$ of $C$ has $|G|/|G_C|$ elements (*used Lemma 1*).
- Each element of $GC$ will appear $|G_C|$ times (*used Lemma 2*).
- Thus each orbit of $X_n$ will appear $|G_C| \cdot |G|/|G_C| = |G|$ many times in our counting. So to find $N(n)$ need to divide the result by $|G|$.
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Proof of Polya’s Theorem

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- For each \(g \in G\), let’s count in how many pairs \((g, C)\) is can appear, i.e. we need to find for each \(g\) how many colorings are invariant under \(g\).
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- Decomposing \(X\) into orbits (=cycles) of \(g\), we see that the color along each cycle must be constant, and that’s the only restriction.

This gives \(n_c(g)\) invariant colorings.

Summing over all \(g \in G\) and dividing by \(|G|\) gives the required formula.
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Weighted Polya theorem

Let \( c_m(g) \) denote the number of cycles of length \( m \) in \( g \in G \) when permuting a finite set \( X \).

**Theorem (Weighted Polya theorem)**

The number of colorings of \( X \) into \( n \) colors with exactly \( r_i \) occurrences of the \( i \)-th color is the coefficient of \( t_1^{r_1} \ldots t_n^{r_n} \) in the polynomial

\[
P(t_1, \ldots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \cdots + t_n^m)^{c_m(g)}
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- The previous formula is obtained by putting $t_1 = \cdots = t_n = 1$.
- What is the number of necklaces with exactly 2 white and 2 black beads? exactly 1 white and 3 black?
(Peg) Solitaire board
A move in the game consists of picking up a marble, and jumping it horizontally or vertically (but not diagonally) over a single marble into a vacant hole, removing the marble that was jumped over.
The goal

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- In other words, are there more winning strategies if we relax the winning condition?
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**Question:** is it easier to win the game finishing at any spot on the board?

- In other words, are there more winning strategies if we relax the winning condition?

- Color spots on the board with **non-trivial** elements of $\mathbb{Z}/2 \times \mathbb{Z}/2$ so that for any 3 consecutive positions (row or column) there are all three elements (let’s call them $f$, $g$, $h$).

(We just re-denote $f = (1, 0)$, $g = (0, 1)$, $h = (1, 1)$.)
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So we should end up with a marble in a position labeled by \( h \) (15 possibilities).
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Thus, if there is a sequence of moves finishing in one spot, then there is a sequence of moves finishing in a symmetric spot.

In other words, there is an action of the group $D_4$ on the set of all possible states of the board.
One more main trick

- **Observation:** allowed moves are invariant under symmetries of the board.
- Thus, if there is a sequence of moves finishing in one spot, then there is a sequence of moves finishing in a symmetric spot.
- In other words, there is an action of the group $D_4$ on the set of all possible states of the board.
- Thus we can only finish in the following spots:
If we finished the game in one of the 4 non-central positions. How could that happen?
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So we might have as well finished in the middle spot.
Generalizations

What about Solitaire games of other shapes?

Figure: French Solitaire
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