Groups of transformations

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Symmetries of a triangle

- Symmetries of the triangle correspond to permutations of vertices $A, B, C$, and vice versa.

- Two types of symmetries: with and without fixed points.
- Symmetries can be composed (i.e. applied one after another).
- Let $s_{AB}, s_{BC}, s_{AC}$ be the symmetries swapping the corresponding vertices. Let $c$ be the symmetry $A \rightarrow B \rightarrow C \rightarrow A$.
- What are their orders, i.e. the number of times you need to compose the symmetry with itself to get the identity symmetry?
- Express all symmetries as compositions of $s_{AB}, s_{BC}$.
- Can you express any symmetry as a composition of $s_{AB}$ and $c$?
- Is such an expression unique?
- Do symmetries $s_{AB}, s_{BC}$ commute?
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- What are the symmetries fixing a point?
- What are some symmetries of order 2 and 4?
- Which symmetries reverse orientation of vertices, and which do not?
Let $c$ be the symmetry $A \to B \to C \to D \to A$.

Exercise: can you express $s_h$, $s_{d1}$ and $s_{d2}$ using $c$ and $s_v$?
Let $c$ be the symmetry $A \to B \to C \to D \to A$.  
$s_v$ be the reflection $A \leftrightarrow B$, $C \leftrightarrow D$;  
$s_h$ be the reflection $A \leftrightarrow D$, $B \leftrightarrow C$;  
$s_{d1}$ be the reflection $B \leftrightarrow D$;  
$s_{d2}$ be the reflection $A \leftrightarrow C$.  

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Suppose you have a mattress.
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You want to make a flipping schedule to prevent your magic mattress from becoming a sagging mattress.
Sagging mattress

Let’s agree, it looks bad (and probably feels not much better).
Mattress moves

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- I
- R
- P
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$$\begin{array}{ccc}
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Y & & \\
\end{array}$$

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Write down the multiplication table for $I, R, P, Y$. 
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You would like to have a single rule of flipping that you can use to achieve every possible mattress position.
Write down the multiplication table for $I, R, P, Y$.
Can you get the desired schedule?
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Symmetries under multiplication form a non-trivial (interesting!) structure.

Not all symmetries commute.

Often the set of symmetries (which can be big!) can be expressed in terms of a very few symmetries, which “generate” this set.
Definition of group of transformation

**Definition**

Let $X$ be a set, and let $G$ be a subset of the set $Bij(X)$ of all bijections $X \rightarrow X$. One says $G$ is a **group** if

1. $G$ is closed under composition;
2. $id \in G$;
3. if $g \in G$, then $g^{-1} \in G$.

**Example**

Symmetries of a triangle, a square and a mattress form a group.
Symmetric group

Take $X = \{1, \ldots, n\}$, and take $G = Bij(X)$ to be the set of all bijections from $X$ to $X$. This group is usually denoted by $S_n$. 

Is $G$ a group?

How many elements does it have?

Definition

The number of elements in a group $G$ is called its order.

For $1 \leq i < j \leq n$ denote by $(ij)$ the permutation swapping $i$ and $j$, and doing nothing to the other elements. Such a permutation is called transposition.

If $j = i + 1$, the transposition $(ij)$ is called a transposition of neighbors.

Prove that any permutation is a composition of transpositions of neighbors.
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It is convenient to denote permutations by

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\sigma = \begin{pmatrix}
1 & 2 & 3 & \ldots & n \\
\sigma(1) & \sigma(2) & \sigma(3) & \ldots & \sigma(n)
\end{pmatrix}
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or simply by

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Find composition $\sigma_2 \circ \sigma_1$ of two permutations

$$\sigma_1 = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 6 & 5 & 2 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 6 & 1 & 5 \end{array} \right)$$
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- Find the inverses of \( \sigma_1 \), \( \sigma_2 \) and \( \sigma_2 \circ \sigma_1 \).
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- Find the inverses of $\sigma_1$, $\sigma_2$ and $\sigma_2 \circ \sigma_1$.

- Verify that $(\sigma_2 \circ \sigma_1)^{-1} = \sigma_1^{-1} \circ \sigma_2^{-1}$. 
Sign of a permutation

Definition

For $\sigma \in S_n$ define $\text{inv}(\sigma)$ to be the number of pairs $(ij)$ such that $i < j$ but $\sigma(i) > \sigma(j)$. This number $\text{inv}(\sigma)$ is called the **number of inversions** of $\sigma$. 

Define the **sign** of $\sigma$ to be $\text{sgn}(\sigma) = (-1)^{\text{inv}(\sigma)}$.

What is the sign of $\sigma = (1 \ 2 \ 3 \ 4 \ 5 \ 6)$?

Prove that for any representation of $\sigma$ as a composition of $N$ transpositions of neighbors, the sign $\text{sgn}(\sigma)$ is $(-1)^N$.

Prove that for two permutations $\sigma, \tau$ we have $\text{sgn}(\sigma \circ \tau) = \text{sgn}(\sigma) \text{sgn}(\tau)$.
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