Vector fields on the plane

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Vector fields

Definition
Suppose at each point of the plane $\mathbb{R}^2$ there is given a vector, so that the coordinates of the vector vary continuously with the point. Then we say we are given a vector field on $\mathbb{R}^2$.

Often convenient to give vector field in the form $f(x, y)\frac{\partial}{\partial x} + g(x, y)\frac{\partial}{\partial y}$. This means that at any point $(x, y)$ we are given the vector with coordinates $(f(x, y), g(x, y))$.

Definition
Points with the zero vector assigned are called singular. We will always assume that our vector fields have finitely many singular points.
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- At any point $x \in C$ dilate the vector $v_x$ at $x$ to a vector $\tilde{v}_x$ at some fixed point on the plane.

This total number of rotations is called the **index** of $C$, denoted $i(C)$. 

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Index of a curve

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- This **total number** of rotations is called the **index** of $C$, denoted $i(C)$. 

**Index of a singular point**

**Definition**

Suppose $z \in \mathbb{R}^2$ is a singular point. Take a closed curve $C$ around $z$ which does not contain any other singular points. Then $i(C)$ is called **index of** $z$ and is denoted by $i(z)$.

**Notice:** this definition makes sense!

**Exercise:** compute indices of singular points of the fields below.
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**Exercise:** compute indices of the following vector fields.
Theorem

Suppose we have a vector field on $\mathbb{R}^2$ and a “nice” curve $C$. Index of a curve $C$ is equal to the sum of indices of singular points inside this curve.
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**Proof:**

![Diagram of a vector field and a curve C]
Index theorem

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Proof:
Very important corollary

Corollary

*If index of a closed curve is not 0, then there is a singular point inside.*
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Application:

**Theorem**

Let \( f : D \to D \) be a continuous map from a disk to itself, such that each point of \( S^1 = \partial D \) is mapped to itself. Then there exists a point \( x \in D \) mapping to the center \( O \) of \( D \).
Theorem

Let \( f : D \rightarrow \mathbb{R}^2 \) be a continuous map from a disk to itself, such that each point of \( S^1 = \partial D \) is mapped to itself. Then there exists a point \( x \in D \) mapping to the center \( O \) of \( D \).

Proof:
Define a vector field on \( D \subset \mathbb{R}^2 \) by \( v_x = f(x) \). On the \( S^1 \subset D \) it will be just \( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \). Index of \( S^1 \) is 1 \( \neq 0 \), so there is a singular point inside. This is what we wanted.
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Any polynomial \( P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0 \) with complex coefficients has a complex root.
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**Proof:**

Consider two vector fields \( v_z = z^n \) and \( w_z = P(z) \).

Let's prove that on a circle \( \{ z \in \mathbb{C} \mid ||z|| = R \} \) for big enough \( R \) holds inequality \( |w_z - v_z| < |v_z| \).

If \( a = \max \{|a_0|, \ldots, |a_{n-1}|\} \), then (for \( R > 1 \))

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|w_z - v_z| = |a_{n-1}z^{n-1} + \cdots + a_0| \leq |a_{n-1}|R^{n-1} + \cdots + |a_0| \leq naR^{n-1}.
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Since \( |v_z| = R^n \) on this circle, then \( |w_z - v_z| < |v_z| \) for \( R > na + 1 \).

So vectors \( v_z \) and \( w_z \) can't point in opposite directions.
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Lemma

Index of the origin $0 \in \mathbb{C}$ with respect to the vector field $v_z = z^n$ equals $n$. 

Proof: Any complex number can be written in the form $z = |z| e^{i\phi}$. Thus can write $v_z$ as $v_z = |z| n e^{i\phi}$. When $\phi$ goes from 0 to $2\pi$, $n\phi$ goes from 0 to $2\pi n$. Done.
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So indices of the circle \( C = \{ z \in \mathbb{C} \mid |z| = R \} \) w.r.t. \( v \) and \( w \) are the same.
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Lemma above gives that index of $C$ w.r.t. $v$ is $n \neq 0$. 

So $w$ has a singular point inside $C$, i.e. the polynomial $P(z)$ has a root inside $C$. 
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