Euler characteristic. Orientatability

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Theorem

Suppose $\Sigma$ is a surface, and $G$ is an embedded graph. Then the Euler characteristic $\chi(\Sigma) := V - E + F$ is correctly defined.

Exercise: compute Euler characteristic of $\mathbb{R}P^2$, $K^2$, $T^2$, $M^2$.

Attaching a Möbius band:
**Question:** How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?
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If $\Sigma'$ is $\Sigma$ with attached Möbius band, then

$$\chi(\Sigma') = \chi(\Sigma) - 1$$
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If $\Sigma'$ is $\Sigma$ with attached Möbius band, then

$$\chi(\Sigma') = \chi(\Sigma) - 1$$

If $\Sigma'$ is $\Sigma$ with attached handle, then

$$\chi(\Sigma') = \chi(\Sigma) - 2$$
Orientable triangulations

Definition
A **triangulation** of a surface $\Sigma$ is an embedding of a graph $G$ into $\Sigma$ such that all faces are triangles.

Definition
A triangulation is **orientable** if all faces can be oriented in a **coherent** way:

![Diagram showing incoherent and coherent orientations](image.png)

Definition
Similarly for any 2-cell decomposition of $\Sigma$. 

Sasha Patotski  (Cornell University)  Euler characteristic. Orientatibility  December 2, 2014  4 / 11
Orientability

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Theorem
The following are equivalent:
1. $\Sigma$ is orientable;
2. any triangulation is orientable;
3. any 2-cell decomposition is orientable.
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Orientability is invariant under barycentric subdivision.
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Orientability is invariant under barycentric subdivision.

Orientability is invariant under coarsening.
Which of the following surfaces are orientable?
Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Proof:
A surface embedded in $\mathbb{R}^n$ is called **compact** if it is closed and bounded.

**Closed:** it contains all its limit points.

**Bounded:** it can be put inside a ball of sufficiently big radius.
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Give example of a closed surface which is not bounded.
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**Closed:** it contains all its limit points.  

**Bounded:** it can be put inside a ball of sufficiently big radius.

Give example of a closed surface which is not bounded.

Give example of a bounded surface which is not closed.
Recall: a surface (in $\mathbb{R}^n$) is a geometric figure that is locally homeomorphic to $\mathbb{R}^2$. 

A surface with boundary is a geometric figure that is locally homeomorphic either to $\mathbb{R}^2$ or to the upper half plane $H^2 = \{ (x, y) \in \mathbb{R}^2 | y \geq 0 \}$.
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Exercise: guess what are some examples.
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Exercise: guess what are some examples.

Surfaces are surfaces with (empty) boundary.
Theorem

**Any compact surface** $\Sigma$ **is determined uniquely up to homeomorphism by the following data:**

$$\chi(\Sigma), \text{ orientability}, \text{ number of boundary components}$$

*In particular, orientable surfaces with no boundary are just spheres with handles. Non-orientable surfaces are just spheres with Möbius bands (more than one).*