Euler characteristic. Orientatibility

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Theorem

Suppose $\Sigma$ is a surface, and $G$ is an embedded graph. Then the Euler characteristic $\chi(\Sigma) := V - E + F$ is correctly defined.

Sketch of a proof:
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Euler characteristic stays the same.
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Euler characteristic is invariant under barycentric subdivision (refinement).
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Euler characteristic is invariant under **coarsening**.
A coarsening of a triangulation $T$ of $\Sigma$ is 2-cell decomposition of $\Sigma$ in which each 2-cell is a union of 2-cells from $T$. 

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Idea: have two triangulations $T_1$ and $T_2$. We want to find a 2-cell decomposition, which is coarsening of some refinement of $T_1$ and is approximating $T_2$. This would finish the proof.

Exercise: compute Euler characteristic of $\mathbb{R}P^2$, $K$, $T^2$, $M^2$.
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**Exercise:** compute Euler characteristic of $\mathbb{R}P^2$, $K^2$, $T^2$, $M^2$. 

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Attaching a Möbius strip

Attaching a handle:

![Diagram](image-url)
Attaching a Möbius strip

Attaching a handle:

Attaching a Möbius band:
Question: How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?
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If $\Sigma$ is $\Sigma''$ with attached handle, then

$$\chi(\Sigma) = \chi(\Sigma'') + 2$$
**Definition**

A **triangulation** of a surface $\Sigma$ is an embedding of a graph $G$ into $\Sigma$ such that all faces are triangles.

**Definition**

A triangulation is **orientable** if all faces can be oriented in a **coherent** way:

![Diagram showing incoherent and coherent orientations]

**Definition**

Similarly for any 2-cell decomposition of $\Sigma$. 
Definition

Surface $\Sigma$ is called **orientable** if there exists orientable triangulation of $\Sigma$. 

Theorem

The following are equivalent:

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2. any triangulation is orientable;
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Easy to see 2 $\iff$ 3 and 2 $\Rightarrow$ 1.

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Which of the following surfaces are orientable?
Lemma about attaching

Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Proof:

(a)  (b)