Embedded graphs

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Embed $K_6$ and $K_7$ into a torus.
2-cell embeddings of graphs

Definition

A 2-cell embedding of a graph $G$ into a surface $\Sigma$ is such an embedding, so that after you remove the graph, the resulting surface will be a disjoint union of discs.

Theorem

Suppose graph $G$ is 2-cell embedded into a sphere with $g$ handles. Then

$$V - E + F = 2 - 2g,$$

where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces.
Theorem

For any graph $G$ there exists a number $g$ so that $G$ can be drawn on a sphere with $g$ handles.

Definition

Define $\gamma(G)$ — minimal number $g$, so that $G$ can be embedded into a sphere of genus $g$.

So $\gamma(K_5) = \gamma(K_{3,3}) = 1$, $\gamma(K_8) \geq 2$, $\gamma(tree) =$?
Theorem

We have $\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}$.

More generally, for polyhedral graphs, $\gamma(G) \geq 1 - \frac{V}{2} + \frac{E}{6}$.

Corollary: $\gamma(K_8) \geq 2$. 
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**Notice:** For a given $g$, any 2-cell decomposition of sphere with $g$ handles has the same number $V - E + F$. 
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**Notice:** 2-cell decomposition makes sense for any surface!
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So far we know that it is correctly defined for spheres with handles.
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Euler characteristic is invariant under coarsening.
Coarsening

Definition
A map of a surface $S$ is a partition of $S$ into properly attached polygons. A coarsening of a triangulation $T$ of $S$ is a map of $S$ in which each polygon is the union of 2-simplices from $T$. 

Euler characteristic is invariant under coarsening.

Idea: have two triangulations $T_1$ and $T_2$. We want to find a map $M$, which is coarsening of some refinement of $T_1$ and is approximating $T_2$. This would finish the proof.

Exercise: compute Euler characteristic of $R P^2$, $K^2$, $T_2$, $S^2$. 

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Exercise: compute Euler characteristic of $\mathbb{RP}^2$, $K^2$, $T^2$, $S^2$. 
Attaching a Möbius strip

Attaching a handle:

![Diagram of attaching a handle to a Möbius strip]
Attaching a Möbius strip

Attaching a handle:

![Diagram of attaching a handle]

Attaching a Möbius band:

![Diagram of attaching a Möbius band]
Lemma about attaching

**Lemma**

*Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.*

**Proof:**

![Diagram](image)

(a) ![Diagram](image)

(b)
Question: How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?