Mapping class groups of surfaces and quantization

Sasha Patotski

Cornell University

ap744@cornell.edu

May 13, 2016
1. Mapping class groups.
2. Quantum representations from skein theory.
3. Quantum representations from geometric quantization.
4. Character varieties as the tensor product of functors.
Let $\Sigma_g$ be a closed compact oriented surface of genus $g$.

**Definition**

The **mapping class group** $\mathcal{M}(\Sigma_g)$ is the group of orientation-preserving diffeomorphisms modulo isotopy:

$$\mathcal{M}_g := \mathcal{M}(\Sigma_g) := \frac{\text{Diff}^+(\Sigma_g)}{\text{Diff}^+_0(\Sigma_g)}$$

**Examples:**

1. $\mathcal{M}(S^2) = \{1\}$;
2. $\mathcal{M}(T) \simeq \text{SL}_2(\mathbb{Z})$. 
Structure of $\mathcal{M}(\Sigma_g)$

**Theorem (Dehn)**

$\mathcal{M}_g$ is generated by Dehn twists along non-separating circles in $\Sigma_g$.

**Fact:** $\mathcal{M}_g$ is a finitely presented group, there are explicit generators and relations.
Definition

Fix $\xi \in \mathbb{C}^\times$. For a 3-manifold $N$, the **skein module** $\mathcal{K}_\xi(N)$ is a $\mathbb{C}$-vector space spanned by the isotopy classes of (framed) links in $N$ modulo the **skein relations**:

$$
\begin{align*}
\begin{array}{c}
\xi^{-1} \bigcirc \bigcirc \\
\xi \bigcirc \bigcirc \\
\end{array} = \begin{array}{c}
\xi^{-2} - \xi^{-2} \\
\end{array}
\end{align*}
$$

**Fact:** $\mathcal{K}_\xi(S^3) \cong \mathbb{C}$. 

Sasha Patotski (Cornell University)
Skein pairing and the action of $\mathcal{M}(\Sigma)$

$\Sigma \hookrightarrow S^3$ such that $S^3 \setminus \Sigma = H \sqcup H'$ is the union of two handlebodies, let $\xi = 4k + 8\sqrt{1}$.

\[\langle -, - \rangle: \mathcal{K}_\xi(H) \times \mathcal{K}_\xi(H') \to \mathcal{K}_\xi(S^3) \simeq \mathbb{C}\]

$\langle -, - \rangle: V_k \times V'_k \to \mathbb{C}$

**Need:** define how Dehn twists acts on $V_k$.

**Fact:** if $\gamma$ bounds a disk in $H$, the Dehn twist $\tau_\gamma$ acts on $\mathcal{K}_\xi(H)$, inducing an action on $V_k$. Similarly, for $\gamma'$ bounding a disk in $H'$, $\tau_{\gamma'}$ acts on $V'_k$.

**Note:** using the pairing, $\tau_{\gamma'}$ also act on $V_k$.

**Fact:** This gives a well-defined action of $\mathcal{M}(\Sigma)$ on $\mathbb{P}(V_k)$.
**Definition**

Call $\mathbb{P}(V_k)$ the **quantum representation of** $\mathcal{M}_g$ of level $k$.

**Theorem (Lickorish)**

Spaces $V_k$ are finite dimensional, and their dimension is

$$d_g(k) := \left( \frac{k + 2}{2} \right)^{g-1} \sum_{j=1}^{k+1} \left( \sin \frac{\pi j}{k + 2} \right)^{2-2g}$$
Let $\Gamma$ be a group, and $G$ a compact Lie group.

Let $\text{Rep}(\Gamma, G)$ be the variety of representations of $\Gamma$ into $G$.

- **Example:** $\text{Rep}(\mathbb{Z}, G) \cong G$;
- **Example:** $\text{Rep}(\mathbb{Z} \times \mathbb{Z}, G) = \{(A, B) \in G \times G \mid AB = BA\}$.

**Note:** $G$ acts on $\text{Rep}(\Gamma, G)$ by conjugation.

The quotient $\mathcal{X}(\Gamma, G) = \text{Rep}(\Gamma, G)/G$ is called the **character variety**.

**Note:** in general $\text{Rep}(\Gamma, G)$ is quite singular, even for “nice” $\Gamma$. 
Let $\Gamma = \pi_1(\Sigma, x_0) \simeq \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$.

Then $\mathcal{X}(\Gamma, G) \simeq \{(A_1, \ldots, B_g) \in G^{2g} \mid [A_1, B_1] \ldots [A_g, B_g] = 1\} / G$.

$\mathcal{X}(\Gamma, G)$ is singular, and let $\mathcal{X}^{\text{reg}} \subset \mathcal{X}(\Gamma, G)$ be the regular part.

**Theorem (Atiyah–Bott)**

For simply-connected $G$, $\mathcal{X}^{\text{reg}}$ has a natural symplectic form $\omega$. The form $\omega$ only depends on the choice of a symmetric form on $g = \text{Lie}(G)$. 
**Fact:** there exists a line bundle $\mathcal{L}$ on $\mathcal{X}^{\text{reg}}$ such that

$$c_1(\mathcal{L}) = [\omega]$$

Pick $\sigma$ — a complex structure on $\Sigma$. Then $\sigma \rightsquigarrow$ complex structure on $\mathcal{X}^{\text{reg}}$. $\mathcal{X} \rightsquigarrow \mathcal{X}_{\sigma}^{\text{reg}}$ a complex manifold, and $\mathcal{L} \rightsquigarrow \mathcal{L}_{\sigma}$ a holomorphic line bundle.

Let $W_{k,\sigma} := H^0(\mathcal{X}^{\text{reg}}, \mathcal{L}_{\sigma}^\otimes k)$ the space of holomorphic sections.

**Theorem**

There is a natural action of the mapping class group $\mathcal{M}(\Sigma)$ on the spaces $\mathbb{P}(W_{k,\sigma})$. Moreover, $W_{k,\sigma}$ are finite dimensional, and for $G = SU(2)$

$$\dim(W_{k,\sigma}) = \left( \frac{k + 2}{2} \right)^{g-1} \frac{k+1}{2} \sum_{j=1}^{\infty} \left( \sin \frac{\pi j}{k + 2} \right)^{2-2g}$$
Main Theorem

Theorem (Andersen–Ueno)

The projective representations $\mathbb{P}(V_k)$ and $\mathbb{P}(W_k)$ of the mapping class group $\mathcal{M}(\sigma)$ are isomorphic.

Remarks:

1. Both constructions can be carried for any compact simply-connected Lie group $G$.
2. In skein theory: choice of $H, H', k$ comes from $\xi = \frac{4k+8}{\sqrt{1}}$.
   In geom.quant.: choice of complex structure $\sigma$, $k$ comes from $k\omega$. 
Let $\mathcal{H}$ be a symmetric monoidal category with

$$\text{Ob}(\mathcal{H}) = \mathbb{N} = \{[0], [1], [2], \ldots\}, \quad [n] \otimes [m] := [n + m],$$

and $\text{Mor}(\mathcal{H})$ generated by

$$m: [2] \to [1], \quad \eta: [0] \to [1], \quad S: [1] \to [1]$$

$$\Delta: [1] \to [2], \quad \varepsilon: [1] \to [0], \quad \tau: [2] \to [2]$$

satisfying the obvious (??) axioms. Graphically,
Note: cocommutative Hopf algebras \( \equiv \) monoidal functors \( F : \mathcal{H} \to \text{Vect} \).
Any functors $F: \mathcal{H} \to \text{Vect}_K$ and $E: \mathcal{H}^{\text{op}} \to \text{Vect}_K$ give $E \otimes_{\mathcal{H}} F \in \text{Vect}_K$.

If $F, E$ are weakly monoidal, then $E \otimes_{\mathcal{H}} F$ is an algebra.

Let $\Gamma$ be a discrete group, and $G$ be an affine algebraic group. Then $K[\Gamma]$ is a cocommutative Hopf algebra, $K(G)$ is a commutative Hopf algebra, and so they define functors

\[
F_\Gamma: \mathcal{H} \to \text{Vect}, \quad [n] \mapsto K[\Gamma]^\otimes n
\]

\[
E_G: \mathcal{H}^{\text{op}} \to \text{Vect}, \quad [n] \mapsto K(G)^\otimes n \cong K(G^n)
\]

\[
E'_G: \mathcal{H}^{\text{op}} \to \text{Vect}, \quad [n] \mapsto K(G^n)^G.
\]

**Theorem (Kassabov–P)**

There are natural algebra isomorphisms

\[
E_G \otimes_{\mathcal{H}} F_\Gamma \cong K(\text{Rep}(\Gamma, G))
\]

\[
E'_G \otimes_{\mathcal{H}} F_\Gamma \cong K(\mathcal{X}(\Gamma, G))
\]
Character variety: \( \mathbb{K}(\mathcal{X}(\Gamma, G)) \simeq E'_G \otimes_{\mathcal{H}} F_{\Gamma} = \bigoplus_{n} \mathbb{K}(G^n)^G \otimes \mathbb{K}[\Gamma]^{\otimes n} \bigg/ \sim \)

Idea: “quantize” \( \mathbb{K}(G), \mathbb{K}[\Gamma] \) and \( \mathcal{H} \).

Assume: \( \Gamma = \pi_1(\Sigma) \)

Replace: \( \mathbb{K}[\Gamma]^{\otimes n} \rightsquigarrow \)

\( \mathbb{K}\{ n\text{-tuples of ribbons in } \Sigma \times (\infty, 0) \text{ with ends in a small fixed disk on } \Sigma \times \{0\} \} \)

Replace: \( \mathbb{K}(G) \rightsquigarrow \mathbb{K}_q(G) \), the corresponding quantum group.

Replace: \( \mathcal{H} \rightsquigarrow \mathcal{R} \) a certain category with objects being slits in an annulus and morphisms being ribbons in the cylinder, connecting the slits.
Morphisms in $\mathcal{R}$ are ribbon analogs of the morphisms in $\mathcal{H}$:

$\Sigma$ gives a functor $F_{\Sigma} : \mathcal{R} \to \text{Vect}_K$, and $K_q(G)$ gives $E_{K_q(G)} : \mathcal{R}^{\text{op}} \to \text{Vect}$.

**Theorem (Kassabov–P)**

$F_{\Sigma} \otimes_{\mathcal{R}} E_{K_q(G)}$ is a (non-commutative) algebra “quantizing” $K(\mathcal{X}(\Gamma, G))$.