An Overview of the Bilinear Hilbert Transform

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A-Exam Presentation
Spring 2018
For $f \in \mathcal{S}(\mathbb{R})$, the **Hilbert transform** is given by:

$$Hf(x) := \lim_{\epsilon \to 0} \int_{|t| > \epsilon} \frac{f(x - t)}{t} \, dt.$$ 

As a **multiplier operator**, it is:

$$\hat{Hf}(\xi) = -\pi i \text{sgn}(\xi) \hat{f}(\xi).$$

The Hilbert transform is an example of a **singular integral operator of Calderón-Zygmund type**. Calderón-Zygmund theory is used to prove the following:
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\]

The Hilbert transform is an example of a **singular integral operator of Calderón-Zygmund type**. Calderón-Zygmund theory is used to prove the following:

**Theorem**

*The Hilbert transform is bounded on \( L^p(\mathbb{R}) \) for every \( 1 < p < \infty \):*

\[
\|Hf\|_{L^p(\mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R})},
\]

for some \( C_p > 0 \).
Calderón-Zygmund Theory Summary

- $L^2 \rightarrow L^2$ bounds follow from Plancherel’s theorem and the properties of the kernel $1/t$.

- $L^1 \rightarrow L^{1,\infty}$ bounds for the Hilbert transform are obtained via the Calderón-Zygmund decomposition.

- $L^p \rightarrow L^p$ bounds, for $1 < p < \infty$, follow from interpolating between the above estimates and duality.
The **bilinear Hilbert transform** in the direction \((\alpha, \beta) \in \mathbb{R}^2\) is defined for \(f, g \in S(\mathbb{R})\) by

\[
BHT_{\alpha,\beta}(f, g)(x) := \lim_{\epsilon \to 0} \int_{|t| > \epsilon} f(x - \alpha t) g(x - \beta t) \frac{dt}{t}.
\]

A. Calderón introduced the \(BHT_{\alpha,\beta}\) while studying the Cauchy integral on Lipschitz curves in 1970.
BHT and History

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**Non-Uniform Bounds** (depending on \((\alpha, \beta)\))

- **M. Lacey, C. Thiele '97**: \(2 < p, q < \infty, 1 < r < 2\)
- **M. Lacey, C. Thiele '99**: \(1 < p, q \leq \infty, 2/3 < r < \infty\) (general case)
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**Uniform Bounds** (independent of \((\alpha, \beta))

- **C. Thiele '02**: weak estimate in \((2, 2, \infty)\)
- **L. Grafakos, X. Li '04**: \(2 < p, q < \infty, 1 < r < 2\)
- **X. Li '06**: \(1 < p, q < 2, 2/3 < r < 1\)
- **R. Oberlin, C. Thiele '11**: expected bounds for Walsh model
The Main Theorem

The result that we study in this talk asserts the following:

Theorem (M. Lacey, C. Thiele, 1999)

The bilinear Hilbert transform maps $L^p \times L^q$ into $L^r$ for any $1 < p, q \leq \infty$ with the property that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $2/3 < r < \infty$. 
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**Figure:** Range for the BHT operator. The plot contains tuples $(1/p, 1/q, 1/r')$, which in our case must lie on the plane $x + y + z = 1$. 
Definition

Let $1 < p, q < \infty$ and $0 < r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. A bilinear operator $T$ is of \textbf{restricted weak type} $(p, q, r)$ if for all measurable sets $E_1, E_2, E$ of finite measure there exists $E' \subset E$ with $|E'| \simeq |E|$ (called a major subset), such that

$$\left| \int_{\mathbb{R}} T(f_1, f_2)(x)f(x) \, dx \right| \lesssim |E_1|^{1/p}|E_2|^{1/q}|E'|^{1/r'}$$

for every $|f_1| \leq \chi_{E_1}, |f_2| \leq \chi_{E_2},$ and $|f| \leq \chi_{E'}$. 
Restricted Weak Type Estimates

Definition

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for every $|f_1| \leq \chi_{E_1}, |f_2| \leq \chi_{E_2}$, and $|f| \leq \chi_{E'}$.

- If $T$ is of restricted weak type $(p, q, r)$, then

$$\| T(f_1, f_2) \|_{r, \infty} \lesssim \| f_1 \|_p \| f_2 \|_q$$

whenever $f_1$ and $f_2$ are as above.
The proof of the main theorem can be reduced to proving the following:

**Theorem**

\[ \text{Fix } \epsilon > 0, \text{ (small). Let } 1 < p < 1 + \epsilon, \ 2 - \epsilon < q < 2, \text{ and such that for } \]
\[ \frac{1}{r} := \frac{1}{p} + \frac{1}{q} \text{ one has } 2/3 < r < 1. \text{ Then the BHT is of restricted weak type } (p, q, r). \]
Figure: The three step interpolation to reduce the main theorem to the theorem on the previous slide. The plot contains tuples $(1/p, 1/q, 1/r')$, which in our case must lie on the plane $x + y + z = 1$. 
Figure: The three step interpolation to reduce the main theorem to the theorem on the previous slide. The plot contains tuples \((\frac{1}{p}, \frac{1}{q}, \frac{1}{r'})\), which in our case must lie on the plane \(x + y + z = 1\).
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Interpolation Details

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Comparison with Bilinear Coifmann-Meyer Operators

- A **bilinear Coifmann-Meyer operator** is an operator of type:
  
  \[
  T(f, g)(x) \leftrightarrow \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x (\xi_1 + \xi_2)} d\xi_1 d\xi_2,
  \]

  where \( |\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^\alpha} \).
A bilinear Coifmann-Meyer operator is an operator of type:

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where \(|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}\). The following bounds hold:

\[ \|T(f, g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q} \text{ for } 1 < p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, 0 < r < \infty. \]
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where \(|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}|\xi|^{|\alpha|} \). The following bounds hold:

\[ ||T(f, g)||_{L^r} \lesssim ||f||_{L^p} ||g||_{L^q} \text{ for } 1 < p, q \leq \infty, \frac{1}{p} + \frac{1}{q} = \frac{1}{r}, 0 < r < \infty. \]

The bilinear Hilbert transform is a bilinear multiplier operator with a more singular multiplier:

\[ BHT(f, g) \rightarrow -i\pi \int_{\mathbb{R}^2} \text{sgn}(\xi_1 - \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x (\xi_1 + \xi_2)} d\xi_1 d\xi_2. \]
Comparing Multiplier Singularities

Bilinear Coifmann-Meyer Operator

Figure: Singularity point, 
\((\xi_1, \xi_2) = (0, 0)\), of multiplier
Comparing Multiplier Singularities

Bilinear Coifmann-Meyer Operator

\[ \xi_2 \]
\[ \xi_1 \]

Figure: Singularity point, \((\xi_1, \xi_2) = (0, 0)\), of multiplier

BHT

\[ \xi_2 \]
\[ \xi_1 \]

Figure: Singularity line, \(\xi_1 = \xi_2\), of BHT multiplier
Discrete Representation of the Multiplier

Bilinear Coifmann-Meyer

Figure: Whitney rectangles with respect to \((0, 0)\)

BHT

Figure: Whitney squares with respect to \(\xi_1 = \xi_2\).
Discrete Representation of the Multiplier

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Figure: Whitney rectangles with respect to $\xi_0, 0$

Figure: Whitney squares with respect to $\xi_1 = \xi_2$. 

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Discrete Representation of the Multiplier

Bilinear Coifmann-Meyer

Figure: Whitney rectangles with respect to \((0, 0)\)

BHT

Figure: Whitney squares with respect to \(\xi_1 = \xi_2\). Giving model:

\[
m(\xi_1, \xi_2) = \sum_{Q} \hat{\phi}_Q(\xi_1) \hat{\phi}_Q(\xi_2) \hat{\phi}_Q(\xi_1 + \xi_2)
\]
We obtain a model operator associated to the BHT given by:

$$BHT_P(f_1, f_2) = \sum_{P \in \mathbb{P}} \frac{1}{|l_P|^{1/2}} \langle f_1, \Phi_{P_1}^1 \rangle \langle f_2, \Phi_{P_2}^2 \rangle \Phi_{P_3}^3.$$ 

Each $P = (P_1, P_2, P_3) \in \mathbb{P}$ is a 3-tuple of tiles in the phase plane.
BHT Model Operator

We obtain a **model operator** associated to the BHT given by:

\[
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\]

Each \( P = (P_1, P_2, P_3) \in P \) is a **3-tuple** of tiles in the **phase plane**.

\[
P_i = l_p \times Q_i
\]

\[
|l_p| \cdot |Q_i| = 1
\]
Properties of Tri-Tiles

- For every dyadic interval $I$, there can be a whole column of tri-tiles $P$ s.t. $I_P = I$.
- The position of $P_1$ or $P_2$ or $P_3$ determines position of the rest.
  - Given location of $P_1/P_2$, then $P_2/P_1$ lies a number of steps away comparable to $C_0$.
  - The frequency coordinate of $P_3$ is essentially a sum of the other two.
- If the frequency intervals of $P_1$ intersect i.e. all contain $\xi_0$, then the frequency intervals of the corresponding $P_2$ tiles are disjoint and \textbf{lacunary} away from $\xi_0$. 
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![Diagram](image_url)
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Diagram:

- X-axis: space
- Y-axis: frequency
- Interval $I$
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![Diagram of frequency and space with intervals $\omega_1$, $\omega_2$, $\omega_1 + \omega_2$ and $\xi_0$.]
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For every dyadic interval $I$, there can be a whole column of tri-tiles $P$ s.t. $l_P = I$.

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Definition

Let $\mathbb{P}$ be any collection of tri-tiles. A subcollection $T \subseteq \mathbb{P}$ is called a **j-tree** provided there exists a tri-tile $P_T$ (called the **top of the tree**) such that

\[
I_P \subset I_{P_T} \quad \text{and} \quad \omega_{P_T,j} \subset 3\omega_{P,j}, \quad \text{for every} \quad P \in T.
\]

**Figure:** $P_1$ tiles in a 1-tree.
\[ L^2 \text{ Size} \]

**Definition**

Let \( \mathcal{P} \) be a finite collection of tri-tiles and let \( f : \mathbb{R} \to \mathbb{C} \). The \( j \)-**size**, for \( j \in \{1, 2, 3\} \), of the sequence \( \langle f, \Phi_{P_j} \rangle_{P \in \mathcal{P}} \) is

\[
\text{size}_{\mathcal{P}} \left( \langle f, \Phi_{P_j} \rangle_{P \in \mathcal{P}} \right) := \sup_{T \subseteq \mathcal{P}} \left( \frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi_{P_j} \rangle|^2 \right)^{1/2},
\]

where \( T \) ranges over all trees in \( \mathcal{P} \) that are \( i \)-trees for \( i \neq j \).
Let $\mathbb{P}$ be a finite collection of tri-tiles and let $f : \mathbb{R} \to \mathbb{C}$. The $j$-size, for $j \in \{1, 2, 3\}$, of the sequence $\langle f, \Phi^j \rangle_{P \in \mathbb{P}}$ is

$$
\text{size}_\mathbb{P} \left( \langle f, \Phi^j \rangle_{P} \right) := \sup_{T \subseteq \mathbb{P}} \left( \frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi^j \rangle|^2 \right)^{1/2},
$$

where $T$ ranges over all trees in $\mathbb{P}$ that are $i-$ trees for $i \neq j$.

Sizes aid in estimating $BHT_{\mathbb{P}}$ when $\mathbb{P}$ consists of a single tree:

$$
\left| \int_{\mathbb{R}} BHT_T(f_1, f_2)(x)f_3(x)dx \right| \leq
$$

$$
\sum_{P \in T} \frac{1}{|I_P|^{1/2}} |\langle f_1, \Phi^1_{P_1} \rangle||\langle f_2, \Phi^2_{P_2} \rangle||\langle f_3, \Phi^3_{P_3} \rangle| \leq |I_T| \prod_{j=1}^{3} \text{size} \left( \langle f_j, \Phi^j \rangle_{P \in T} \right)
$$
Let $1 \leq i \leq 3$. A finite sequence of trees $T_1, \ldots, T_M$ is a chain of strongly i-disjoint trees provided that: they are pairwise disjoint and

- If $P \in T_{\ell_1}, P' \in T_{\ell_2} (\ell_1 \neq \ell_2)$ with $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \emptyset$ then
  
  \[ |\omega_{P_i}| \leq |\omega_{P'_i}| \implies I_P \cap I_{T_{\ell_1}} = \emptyset \text{ and} \]
  
  \[ |\omega_{P'_i}| < |\omega_{P_i}| \implies I_{P'} \cap I_{T_{\ell_2}} = \emptyset. \]
Let $1 \leq i \leq 3$. A finite sequence of trees $T_1, \ldots, T_M$ is a **chain of strongly $i$-disjoint trees** provided that: they are pairwise disjoint and

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  $|\omega_{P_i}| \leq |\omega_{P'_i}| \quad \implies \quad I_{P'} \cap I_{T_{\ell_1}} = \emptyset$ and

  
  
  $|\omega_{P'_i}| < |\omega_{P_i}| \quad \implies \quad I_P \cap I_{T_{\ell_2}} = \emptyset$.  

\textbf{Not a Disjoint Chain}
The **j-energy** of the sequence $\langle f, \Phi^j_{P_j} \rangle_{P \in \mathcal{P}}$ is

$$
\text{energy} \left( \langle f, \Phi^j_{P_j} \rangle_{P} \right) := \sup_{n \in \mathbb{Z}} \sup_{\mathcal{T}} 2^n \left( \sum_{T \in \mathcal{T}} |I_T| \right)^{1/2},
$$

$\mathcal{T}$ ranges over chains of strongly $j-$ disjoint trees (which are $i$-trees for some $i \neq j$) having the property that

$$
\text{size} \left( \langle f, \Phi^j_{P_j} \rangle_{P} \right) \approx 2^n.
$$

- The energy aids in summing the estimates obtained on each individual tree.
The following theorem provides a way of estimating a generic trilinear form associated with $BHT_{\mathbb{P}}(f_1, f_2)$. First we write:

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) := \int_{\mathbb{R}} BHT_{\mathbb{P}}(f_1, f_2)(x) f_3(x) dx.$$ 

**Theorem (size-energy estimate)**

*Let $\mathbb{P}$ be a finite collection of tri-tiles. Then*

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} (\text{size}(\langle f, \Phi_{P_j}^j \rangle_{\mathbb{P}}))^{\theta_j} (\text{energy}(\langle f, \Phi_{P_j}^j \rangle_{\mathbb{P}}))^{1-\theta_j}$$

*for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.***
The following is a key ingredient in the proof of the size-energy duality theorem.

**Proposition (stopping-time decomposition)**

Let $j \in \{1, 2, 3\}$. For any $\mathbb{P}' \subset \mathbb{P}$ and any $n \in \mathbb{Z}$ such that

\[
\text{size}(\{f, \Phi_{P_j}^{P} \}_{P \in \mathbb{P}'}) \leq 2^{-n} \text{energy} \left( \{f, \Phi_{P_j}^{P} \}_{P \in \mathbb{P}} \right),
\]

One can decompose $\mathbb{P}' = \mathbb{P}^- \cup \mathbb{P}^+$ in such a way that

\[
\text{size}(\{f, \Phi_{P_j}^{P} \}_{P \in \mathbb{P}^-}) \leq 2^{-n-1} \text{energy} \left( \{f, \Phi_{P_j}^{P} \}_{P \in \mathbb{P}} \right)
\]

and $\mathbb{P}^+$ can be written as a disjoint union of trees $T \in \mathcal{T}$ such that

\[
\sum_{T \in \mathcal{T}} |l_T| \lesssim 2^{2n}.
\]
Proof.

(WLOG take \( j = 2 \)) Consider all \( i \)-trees \( T \) (\( i \neq 2 \)) that are upward 2- trees rooted at \( P_T \) and satisfy:

\[
\left( \frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi^j_P \rangle|^2 \right)^{1/2} > 2^{-n-1} \text{energy} \left( \langle f, \Phi^j_P \rangle_{P \in \mathbb{P}} \right).
\]

If there are no such trees, terminate algorithm.
Proof.

(WLOG take $j=2$) Consider all $i$-trees $T$ ($i \neq 2$) that are upward 2–trees rooted at $P_T$ and satisfy:

$$\left( \frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi^j_{P} \rangle|^2 \right)^{1/2} > 2^{-n-1} \text{energy} \left( \langle f, \Phi^j_P \rangle_{P \in \mathcal{P}} \right).$$

If there are no such trees, terminate algorithm. Otherwise, choose a maximal $T$ whose center $\xi_{T,i}$ of $\omega_{P_T,i}$ is largest.
Proof of Stopping-Time Decomposition

**Proof.**

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If there are no such trees, terminate algorithm. Otherwise, choose a maximal $T$ whose center $\xi_{T,i}$ of $\omega_{P_T,i}$ is largest. Remove $T$ and $\tilde{T}$ from $\mathbb{P}'$ and place into $\mathbb{P}^+$ where:

$$
\tilde{T} := \{ P \in \mathbb{P}' \setminus T | l_P \subseteq l_{P_T}, \omega_{P_T,2} \subseteq 3 \omega_{P,2} \}.
$$
Proof of Stopping-Time Decomposition

Proof.

(WLOG take \( j = 2 \)) Consider all i-trees \( T \) \( (i \neq 2) \) that are upward 2 trees rooted at \( P_T \) and satisfy:

\[
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If there are no such trees, terminate algorithm. Otherwise, choose a maximal \( T \) whose center \( \xi_{T,i} \) of \( \omega_{P_T,i} \) is largest. Remove \( T \) and \( \tilde{T} \) from \( \mathbb{P}' \) and place into \( \mathbb{P}^+ \) where:

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\]

Continue until algorithm terminates. Trees \( T_1, T_2, ..., T_M \) form a chain of strongly 2 disjoint trees.
Proof.

(WLOG take $j=2$) Consider all $i$-trees $T$ ($i \neq 2$) that are upward 2- trees rooted at $P_T$ and satisfy:

\[
\left( \frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi_P^j \rangle|^2 \right)^{1/2} > 2^{-n-1} \text{energy} \left( \langle f, \Phi_P^j \rangle_{P \in \mathbb{P}} \right).
\]

If there are no such trees, terminate algorithm. Otherwise, choose a maximal $T$ whose center $\xi_{T,i}$ of $\omega_{P_T,i}$ is largest. Remove $T$ and $\tilde{T}$ from $\mathbb{P}'$ and place into $\mathbb{P}^+$ where:

\[
\tilde{T} := \{ P \in \mathbb{P}' \mid T \cap I_P \subseteq I_{P_T}, \omega_{P_T,2} \subseteq 3\omega_{P,2} \}.
\]

Continue until algorithm terminates. Trees $T_1, T_2, ..., T_M$ form a chain of strongly 2- disjoint trees. Repeat for downward 2- trees.
Justifying that $T_1, T_2, \ldots, T_M$ form a chain of strongly $j$–disjoint trees

- Assume $T_s, T_{s'}$ do not satisfy the strongly 2-disjointness property.

- So, there are $P \in T_s, P' \in T_{s'}$ with $|\omega_{P_2}| < |\omega_{P_1'}|$ and $l_{P'} \subseteq I_{T_s}$.

- But $|\omega_{P_2}| < |\omega_{P_1'}|$ implies $\xi_{P_{T_{s'}}, i} < \xi_{P_{T_s}, i}$.

- So, $T_s$ is selected before $T_{s'}$.

- Hence, the tri-tile $P'$ was removed in $\widetilde{T}_s$, contradicting that $P' \in T_{s'}$. 

\[ \xi \]

\[ \xi_{T_s, 1} \]

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\[ x \]
Justifying that $T_1, T_2, \ldots, T_M$ form a chain of strongly $j$–disjoint trees

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Justifying that \( T_1, T_2, \ldots, T_M \) form a chain of strongly \( j \)-disjoint trees

- Assume \( T_s, T_s' \) do not satisfy the strongly 2-disjointness property.

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- So, \( T_s \) is selected before \( T_s' \).

- Hence, the tri-tile \( P' \) was removed in \( \tilde{T}_s \), contradicting that \( P' \in T_s' \).
Iterated Stopping-Time Decomposition

\[ \mathbb{P} - \text{size}(\langle f, \Phi_{P_i} \rangle_{P \in \mathbb{P}}) \leq 2^{-k} E_i \quad \text{for} \quad i \in \{1, 2, 3\} \]

\[ \mathbb{P}^+_1 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^k} \]
\[ \mathbb{P}^-_1, \mathbb{P}^+_2, \mathbb{P}^-_3 \]

\[ \mathbb{P}^-_1, k+1 - \text{size}(\langle f, \Phi_{P_1} \rangle_{P \in \mathbb{P}^-_1, k+1}) \leq 2^{-(k+1)} E_1 \]

\[ \mathbb{P}^+_2 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^k} \]
\[ \mathbb{P}^-_2, \mathbb{P}^+_3 \]

\[ \mathbb{P}^-_2, k+1 - \text{size}(\langle f, \Phi_{P_2} \rangle_{P \in \mathbb{P}^-_2, k+1}) \leq 2^{-(k+1)} E_2 \]

\[ \mathbb{P}^+_3 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^k} \]
\[ \mathbb{P}^-_3, \mathbb{P}^+_1 \]

\[ \mathbb{P}^-_3, k+1 - \text{size}(\langle f, \Phi_{P_3} \rangle_{P \in \mathbb{P}^-_3, k+1}) \leq 2^{-(k+1)} E_3 \]

\[ \mathbb{P}^+_1, k+1 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^{k+1}} \]
\[ \mathbb{P}^-_1, k+2 - \text{size}(\langle f, \Phi_{P_1} \rangle_{P \in \mathbb{P}^-_1, k+2}) \leq 2^{-(k+2)} E_1 \]

\[ \mathbb{P}^+_2, k+1 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^{k+1}} \]
\[ \mathbb{P}^-_2, k+2 - \text{size}(\langle f, \Phi_{P_2} \rangle_{P \in \mathbb{P}^-_2, k+2}) \leq 2^{-(k+2)} E_2 \]

\[ \mathbb{P}^+_3, k+1 \]
\[ \sum_{T \in T} |I_T| \lesssim 2^{2^{k+1}} \]
\[ \mathbb{P}^-_3, k+2 - \text{size}(\langle f, \Phi_{P_3} \rangle_{P \in \mathbb{P}^-_3, k+2}) \leq 2^{-(k+2)} E_3 \]

\[ \quad \cdots \]
Corollary

Let $\mathcal{P}$ be a collection of tri-tiles. One can split $\mathcal{P}$ as

$$\mathcal{P} = \bigcup_{k \in \mathbb{Z}} \mathcal{P}_k,$$

where for $k \in \mathbb{Z}$ we have

$$\text{size} \left( \left\langle f, \Phi_{P_j}^j \right\rangle_{P \in \mathcal{P}_k} \right) \leq \min(2^{-k} E_j, S_j), \quad \text{for every } j = 1, 2, 3.$$

Also, one can cover $\mathcal{P}_k$ by a collection of trees $T \in T_k$ for which

$$\sum_{T \in T_k} |I_T| \lesssim 2^{2k}.$$
Iterated Stopping-Time to Deduce Size-Energy Estimate

- $\mathcal{P} = \bigcup_{k \in \mathbb{Z}} \mathcal{P}_k^+$
- $\mathcal{P}_k$ covered by trees $T \in \mathbb{T}_k$ for which $\sum_{T \in \mathbb{T}_k} |I_T| \lesssim 2^{2^k}$
- $S_{\mathcal{P}_k^+}^j \lesssim 2^{-k} E_{\mathcal{P}}^j$

\[
\left| \Lambda_{\mathcal{P}} \left( \frac{f_1}{E_1}, \frac{f_2}{E_2}, \frac{f_3}{E_3} \right) \right| = \sum_{k \in \mathbb{Z}} \left( \prod_{j=1}^{3} \text{size} \left( \left\langle \frac{f_j}{E_j}, \Phi_P^j \right\rangle_{P \in \mathcal{P}_k} \right) \right) \sum_{T \in \mathbb{T}_k} |I_T| \\
\lesssim \sum_{k \in \mathbb{Z}} \left( \prod_{j=1}^{3} \text{size} \left( \left\langle \frac{f_j}{E_j}, \Phi_P^j \right\rangle_{P \in \mathcal{P}_k} \right) \right) 2^{2^k} \\
\lesssim \left( \frac{S_1}{E_1} \right)^{\theta_1} \left( \frac{S_2}{E_2} \right)^{\theta_2} \left( \frac{S_3}{E_3} \right)^{\theta_3} .
\]
Lemma (Maximal Operator Bound)

Let \( j \in \{1, 2, 3\} \), then for every \( f \in S(\mathbb{R}) \) one has

\[
\text{size} \left( \langle f, \Phi_{P_j} \rangle_P \right) \lesssim \sup_{P \in \mathcal{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f(x)| \cdot \tilde{\chi}_{I_P}(x) \, dx.
\]
Bound on the Size

Lemma (Maximal Operator Bound)

Let $j \in \{1, 2, 3\}$, then for every $f \in \mathcal{S}(\mathbb{R})$ one has

$$\text{size} \left( \langle f, \Phi^j_{P_j} \rangle \right) \lesssim \sup_{P \in \mathcal{P}} \frac{1}{|P|} \int_{\mathbb{R}} |f(x)| \cdot \tilde{\chi}_P(x) \, dx.$$  

- John-Nirenberg Inequality:

$$\text{size}^j_{\mathcal{P}} := \sup_{T \subset \mathcal{P}} \frac{1}{|T|^{1/2}} \left\| \left( \sum_{P \in T} \frac{|\langle f, \Phi^j_{P_j} \rangle|^2}{|P|} \chi_P \right)^{1/2} \right\|_2 \quad \lesssim \quad \sup_{T \subset \mathcal{P}} \frac{1}{|T|} \left\| \left( \sum_{P \in T} \frac{|\langle f, \Phi^j_{P_j} \rangle|^2}{|P|} \chi_P \right)^{1/2} \right\|_{1, \infty}.$$
Lemma (Bessel Inequality)

Let $j \in \{1, 2, 3\}$ and $f \in L^2(\mathbb{R})$. Then

$$\text{Energy}\left( \langle f, \Phi^j_P \rangle \right) \lesssim ||f||_2.$$
Bound on the Energy

Lemma (Bessel Inequality)

Let \( j \in \{1, 2, 3\} \) and \( f \in L^2(\mathbb{R}) \). Then

\[
\text{Energy} \left( \langle f, \Phi^j_{P_j} \rangle_P \right) \lesssim \|f\|_2.
\]

- One invokes **strong j-disjointness** of trees \( T \in \mathcal{T} \) and **almost orthogonality** of the corresponding wave packets \( \{\Phi^j_{P_j}\}_{P \in T} : 

\[
E \left( \langle f, \Phi^j_{P_j} \rangle_P \right)^2 = 2^n \left( \sum_{T \in \mathcal{T}} |I_T| \right) \lesssim 2^n 2^{-n} \left( \sum_{T \in \mathcal{T}} \left( \sum_{P \in T} |\langle f, \Phi^j_{P_j} \rangle|^2 \right) \right)
\]

\[
= \left| \left\| \sum_T \sum_{P \in T} \langle f, \Phi_{P_j} \rangle \Phi_{P_j} f \right\| \right| \lesssim \|f\|_2 \left\| \sum_T \sum_{P \in T} \langle f, \Phi_{P_j} \rangle \Phi_{P_j} \right\|_2.
\]
Fix measurable sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}$ of finite measure. **Our goal is to construct a subset $\mathcal{E}' \subseteq \mathcal{E}$ with $|\mathcal{E}'| \simeq |\mathcal{E}|$ and such that**

\[
\left| \sum_{P \in \mathcal{P}} \frac{1}{|I_P|^{1/2}} \langle f_1, \Phi_{P_1}^1 \times f_2, \Phi_{P_2}^2 \times f_3, \Phi_{P_3}^3 \rangle \right| \lesssim |\mathcal{E}_1|^{1/p} |\mathcal{E}_2|^{1/q} |\mathcal{E}'|^{1/r'} \tag{1}
\]

For every $|f_1| \leq \chi_{\mathcal{E}_1}, |f_2| \leq \chi_{\mathcal{E}_2},$ and $|f| \leq \chi_{\mathcal{E}'}$. 
Proof of Main Theorem (Continued)

Define first an **exceptional set**

\[ \Omega := \{ x : Mf_1(x) > C|\mathcal{E}_1| \} \cup \{ x : Mf_2(x) > C|\mathcal{E}_2| \}, \]

where \( M \) is the usual Hardy-Littlewood maximal operator.
Proof of Main Theorem (Continued)

- Define first an **exceptional set**

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where \( M \) is the usual Hardy-Littlewood maximal operator.

- Set \( \mathcal{E}' := \mathcal{E}\setminus\Omega \). It satisfies \( |\mathcal{E}'| \asymp |\mathcal{E}| \) if \( C \) is sufficiently large enough.
Define first an **exceptional set**

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where \( M \) is the usual Hardy-Littlewood maximal operator.

Set \( \mathcal{E}' := \mathcal{E}\setminus\Omega \). It satisfies \( |\mathcal{E}'| \sim |\mathcal{E}| \) if \( C \) is sufficiently large enough.

To be able to estimate we split our collection of tri-tiles \( \mathcal{P} \) as follows:

\[ \mathcal{P} = \bigcup_{d \geq 0} \mathcal{P}_d, \]

where \( \mathcal{P}_d \) contains all the tri-tiles in \( \mathcal{P} \) having the property that

\[ 2^d \leq 1 + \frac{\text{dist}(I_p, \Omega^c)}{|I_p|} < 2^{d+1}. \]
Proof of Main Theorem (Continued)

So, one has

$$\left| \Lambda_{\mathbb{P}}(f_1, f_2, f_3) \right| \leq \sum_{d=0}^{\infty} \left| \Lambda_{\mathbb{P}_d}(f_1, f_2, f_3) \right|$$

$$\leq \sum_{d=0}^{\infty} \left( \prod_{j=1}^{3} \left( \text{size} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left( \text{energy} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. 
Proof of Main Theorem (Continued)

So, one has

$$|\Lambda_\mathbb{P}(f_1, f_2, f_3)| \leq \sum_{d=0}^{\infty} |\Lambda_{\mathbb{P}_d}(f_1, f_2, f_3)|$$

$$\lesssim \sum_{d=0}^{\infty} \left( \prod_{j=1}^{3} \left( \text{size} \left( \langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left( \text{energy} \left( \langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)$$

for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

- **Size Lemma (Maximal Function Bound):**

  $$\text{size} \left( \langle f_i, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \lesssim \sup_{P \in \mathbb{P}_d} \frac{1}{|P|} \int f_i \tilde{\chi}_P M dx \implies \begin{cases} S_i \lesssim 2^d |\mathcal{E}_i| \\ S_3 \lesssim 2^{-Md} |\mathcal{E}'| \end{cases}$$
Proof of Main Theorem (Continued)

So, one has

\[ |\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \leq \sum_{d=0}^{\infty} |\Lambda_{\mathbb{P}_d}(f_1, f_2, f_3)| \]

\[ \leq \sum_{d=0}^{\infty} \left( \prod_{j=1}^{3} \left( \text{size} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left( \text{energy} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \]

for any \( 0 \leq \theta_1, \theta_2, \theta_3 < 1 \) with \( \theta_1 + \theta_2 + \theta_3 = 1 \).

- **Size Lemma (Maximal Function Bound):**

  \[
  \text{size} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \lesssim \sup_{P \in \mathbb{P}_d} \frac{1}{|l_P|} \int f_i \tilde{\chi}_{l_P}^M dx \implies \begin{cases} S_i \lesssim 2^d |\mathcal{E}_i| \\ S_3 \lesssim 2^{-Md} |\mathcal{E}'| \end{cases} \quad i=1,2
  \]

- **Energy Lemma (Bessel Inequality):**

  \[
  \text{energy} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \lesssim \|f_i\|_2 \implies \text{energy} \left( \langle f, \Phi_{P_j}^j \rangle_{P \in \mathbb{P}_d} \right) \lesssim |\mathcal{E}_i|^{1/2}.
  \]
So,

\[
\sum_{d=0}^{\infty} \left( \prod_{j=1}^{\text{size} \left( \langle f, \Phi_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \theta_j \left( \text{energy} \left( \langle f, \Phi_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)
\]

\[
\sum_{d=0}^{\infty} \left( 2^d |\mathcal{E}_1| \right)^{\theta_1} |\mathcal{E}_1|^{(1-\theta_1)/2} \left( 2^d |\mathcal{E}_2| \right)^{\theta_2} |\mathcal{E}_2|^{(1-\theta_2)/2} 2^{-Md \theta_3}
\]

\[
= \sum_{d=0}^{\infty} 2^{d(\theta_1+\theta_2-M \theta_3)} |\mathcal{E}_1|^{(1+\theta_1)/2} |\mathcal{E}_2|^{(1+\theta_2)/2} \leq |\mathcal{E}_1|^{(1+\theta_1)/2} |\mathcal{E}_2|^{(1+\theta_2)/2}
\]

Setting \(1/p := (1 + \theta_1)/2\) and \(1/q = (1 + \theta_2)/2\) completes the proof.


Thank you for your attention!