Lec 9: Solving Linear Systems

Solving linear system \( A\bar{x} = \bar{b} \) can be carried out in two steps:

- Producing a row echelon form \([A' | \bar{b}]\) of the augmented matrix \([A | \bar{b}]\).
- Solving the linear system \( A' \bar{x} = \bar{b} \) by back substitution.

This method is called Gaussian elimination. If in the first step one gets the reduced row echelon form of \([A | \bar{b}]\), one uses Gauss-Jordan reduction method.

**Example.** Solve the linear system

\[
\begin{align*}
  x - y + 2z &= 0 \\
  3x - z &= 1 \\
 -2x + 4y + z &= -2
\end{align*}
\]

We will use the Gaussian elimination. That is, we will transform the augmented matrix

\[
B = \begin{bmatrix}
  1 & -1 & 2 & 0 \\ 
  3 & 0 & -1 & 1 \\ 
 -2 & 4 & 1 & -2 
\end{bmatrix}
\]

to a row echelon form. We have:

\[
C = B_{r_2 - 3r_1 \rightarrow r_2} = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  0 & 3 & -7 & 1 \\
 -2 & 4 & 1 & -2 
\end{bmatrix},
\]

\[
D = C_{r_3 + 2r_1 \rightarrow r_3} = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  0 & 3 & -7 & 1 \\
  0 & 2 & 5 & -2 
\end{bmatrix},
\]

\[
E = D_{3r_2 \rightarrow r_2} = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  0 & 1 & -\frac{7}{3} & \frac{1}{3} \\
  0 & 2 & 5 & -2 
\end{bmatrix},
\]

\[
F = E_{r_3 - 2r_1 \rightarrow r_3} = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  0 & 1 & -\frac{7}{3} & \frac{1}{3} \\
  0 & 0 & \frac{5}{3} & -\frac{7}{3} 
\end{bmatrix},
\]

\[
G = F_{29 \rightarrow r_3} \rightarrow r_3 = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  0 & 1 & -\frac{7}{9} & \frac{1}{29} \\
  0 & 0 & 1 & -\frac{8}{29} 
\end{bmatrix}.
\]

Matrix \( G \) has a row echelon form. Now we have to solve the linear system with the augmented matrix \( G \):

\[
\begin{align*}
  x - y + 2z &= 0 \\
  y - \frac{7}{3}z &= \frac{1}{3} \\
  z &= -\frac{8}{29}
\end{align*}
\]

The bottom equation tells us what \( z \) is. The second one yields \( y = \frac{7}{3}z + \frac{1}{3} = \frac{7}{3}(-\frac{8}{29}) + \frac{1}{3} = -\frac{9}{29} \). Finally, from the top equation we have \( x = y - 2z = -\frac{9}{29} - 2(-\frac{8}{29}) = \frac{7}{29} \).

Thus, system (1) has a unique solution \( x = \frac{7}{29}, y = -\frac{9}{29}, z = -\frac{8}{29} \).
One may wonder why the solutions of systems (1) and (2) are the same. Note that matrix $G$ is produced from matrix $B$ by a sequence of elementary row transformations. These transformations correspond to: (i) interchanging equations; (ii) multiplication of an equation by a nonzero scalar; (iii) adding a multiple of an equation to another one. Each of these operations transforms a linear system to an equivalent one (i.e. with the same solutions). Hence so does a sequence of such operations. This explains why the system (1) with matrix $B$ has the same solutions as the system (2) with matrix $G$.

In our example we could have proceeded transforming $G$ to a reduced echelon form. [That is solving system (1) by the method of Gauss-Jordan reduction.] We have:

$$ H = G_{r_2+\frac{7}{3}r_3\to r_2} = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 0 & -\frac{2}{29} \\ 0 & 0 & 1 & -\frac{9}{29} \end{bmatrix}, \quad I = G_{r_1-2r_3\to r_1} = \begin{bmatrix} 1 & -1 & 0 & \frac{16}{29} \\ 0 & 1 & 0 & \frac{1}{29} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix}, $$

$$ J = I_{r_1+r_2\to r_1} = \begin{bmatrix} 1 & 0 & 0 & \frac{7}{29} \\ 0 & 1 & 0 & \frac{1}{29} \\ 0 & 0 & 1 & -\frac{8}{29} \end{bmatrix}. $$

Matrix $J$ is in a reduced row echelon form, and the corresponding linear system is:

$$ x = \frac{7}{29}, \quad y = -\frac{1}{29}, \quad z = -\frac{8}{29}. \quad (3) $$

This system is trivial, and we don’t need to perform any other operations. In fact, this is a general observation. In Gauss-Jordan method we don’t need to solve a system by the back substitution because we were solving it implicitly when getting matrices from an echelon form to a reduced echelon form. For example, a linear system with an augmented matrix

$$ \begin{bmatrix} 1 & 0 & 3 & 0 & 5 \\ 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad (4) $$

[which is in a reduced row echelon form] has the solution $x_1 = 5 - 3x_3, x_2 = 1 - 4x_3, x_4 = 2$. [If the variables are $x_1, x_2, x_3, x_4$.] Here $x_3$ can be any number, so the system has infinitely many solutions. If the bottom row of matrix (4) was $[0 \ 0 \ 0 \ 0 \ 2]$, then the system would be inconsistent (i.e. no solutions). Generally, given a system with an augmented matrix $B = [A|\vec{b}]$ in a reduced row echelon form, to get solution, we have to express variables corresponding to leading ones of $B$ through other variables and entries of vector $\vec{b}$. 

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