Lec 26: Transition matrix.

Let $V$ be an $n$-dimensional vector space and $S = \{v_1, \ldots, v_n\}, T = \{w_1, \ldots, w_n\}$ its two bases. The transition matrix $P_{S \to T}$ from $T$ to $S$ is $n \times n$ matrix which columns are coordinates of $w_j$ in basis $S$:

$$P_{S \to T} = [w_1]_S [w_2]_S \ldots [w_n]_S.$$

As we will see, by means of this matrix one can transform coordinates of a vector in basis $T$ to coordinates in $S$. But before the theorem, let’s look at examples of finding $P_{S \to T}$.

**Example 1.** $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$ — standard basis, $T = \{w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$. Then

$$P_{S \to T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}.$$

(e. g. $w_3 = 2e_2 + e_3$)

**Example 2.** $V = \mathbb{R}^3$, $S = \{v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}, T = \{w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, w_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$. To find the coordinates $x_1, x_2, x_3$ of $w_1$ in basis $S$, we have to solve the linear system:

$$x_1v_1 + x_2v_2 + x_3v_3 = w_1.$$

Its augmented matrix is $[v_1 \ v_2 \ v_3 | w_1]$. The RREF will be $[I_3 | \mathbf{x}]$ for some $\mathbf{x}$ (the matrix $[v_1 \ v_2 \ v_3]$ has nonzero det, so its RREF is the identity matrix). Clearly, then $\mathbf{x}$ will be the solution. Similarly, to find the coordinates of $w_2, w_3$ in $S$, we have to solve linear systems with augmented matrices $[v_1 \ v_2 \ v_3 | w_2]$, $[v_1 \ v_2 \ v_3 | w_3]$. Hence we can do it at once by producing the RREF for the partitioned matrix

$$[v_1 \ v_2 \ v_3 | w_1 | w_2 | w_3] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 2 & 1 \end{bmatrix}$$

which one can find is

$$\begin{bmatrix} 1 & 0 & 0 & | & 1.5 & | & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 & | & 1 & | & -1 \\ 0 & 0 & 1 & | & -4.5 & | & 2 & | & -2 \end{bmatrix}.$$
. Then the last three column are exactly \([w_1]_S, [w_2]_S\) and \([w_3]_S\). So, the transition matrix is

\[
P_{S\rightarrow T} = \begin{bmatrix} 1.5 & 0 & 1 \\ -2 & 1 & -1 \\ -4.5 & 2 & -2 \end{bmatrix}.
\]

**Theorem 0.1.** For any vector \(v\) in \(V\) we have \([v]_S = P_{S\rightarrow T}[v]_T\).

**Proof.** Let \(v = c_1w_1 + c_2w_2 + \cdots + c_n w_n\) (in other words, \([v]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\)). Then \([v]_S = [c_1]_S [c_2]_S \cdots [c_n]_S = [c_1]_S [c_2]_S \cdots [c_n]_S = (P_{S\rightarrow T}[v]_T).

In example 2, if \(v = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}\), then one can show \(v = 2w_2 + w_3\), or equivalently,

\([v]_T = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\). Then by Theorem we can find its coordinates in \(S\):

\([v]_S = P_{S\rightarrow T}[v]_T\).

Hence \(v = v_1 + v_2 + 2v_3\).

Point out some properties of \(P_{S\rightarrow T}\):

- \(P_{S\rightarrow S} = I_n\) - identity matrix (why?).

- If \(R, S, T\) are bases in \(V\), then \(P_{R\rightarrow S}P_{S\rightarrow T} = P_{R\rightarrow T}\). Indeed for any vector \(v\) in \(V\) we have by the Theorem: \([v]_R = P_{R\rightarrow S}[v]_S = P_{R\rightarrow S}(P_{S\rightarrow T}[v]_T) = (P_{R\rightarrow S}P_{S\rightarrow T})[v]_T\). On the other hand, we know \([v]_R = P_{R\rightarrow T}[v]_T\). Then \((P_{R\rightarrow S}P_{S\rightarrow T})[v]_T = P_{R\rightarrow T}[v]_T\). Since this holds for any \(v\) (hence any \([v]_T\)), the matrices on the left coincide (why?): \(P_{R\rightarrow S}P_{S\rightarrow T} = P_{R\rightarrow T}\).

- The transition matrix from \(T\) to \(S\) is invertible and its inverse is the transition matrix from \(S\) to \(T\): \(P_{S\rightarrow T}^{-1} = P_{T\rightarrow S}\). This follows from the previous properties, if we take \(R = S\).

In example 2 we could compute \(P_{S\rightarrow T}\) using the properties. Denote by \(St\) the standard basis in \(\mathbb{R}^3\). Then \(P_{S\rightarrow T} = P_{S\rightarrow St}P_{St\rightarrow T} = P_{St\rightarrow S}P_{S\rightarrow T}\). The transition
matrices to the standard basis are obvious (example 1), so the only nontrivial thing
is to find the inverse of the first matrix (do it!). We have
\[
P_{S\to T} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -5 & -0.5 \\ -1 & -1 & 1 \\ -1.5 & 3 & 2.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 & 1 \\ -2 & 1 & -1 \\ -4.5 & 2 & -2 \end{bmatrix},
\]

exactly the matrix we got before.

**Example 3.** Let \( V = \text{Pol}(1), S = \{v_1 = t, v_2 = t - 3\}, T = \{w_1 = t - 1, w_2 = t + 1\}. \)

For the standard basis \( St = \{t, 1\} \) we have
\[
P_{St\to S} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}, \quad P_{St\to T} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.
\]

As in example 2, to find \( P_{S\to T} \), we have to produce RREF for the partitioned matrix
\[
[P_{St\to S}|P_{St\to T}] = \begin{bmatrix} 1 & 1 & | & 1 & 1 \\ 0 & -3 & | & -1 & 1 \end{bmatrix},
\]

which is
\[
\begin{bmatrix} 1 & 0 & | & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & | & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.
\]

Then take the matrix on the right:
\[
P_{S\to T} = \frac{1}{3} \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}.
\]

Another method of finding the transition matrix is \( P_{S\to T} = P_{St\to S}^{-1} P_{St\to T} \) proved before the example. Verify that this way we get the same matrix. Note that \( 5t - 1 = 3(t - 1) + 2(t + 1) = 3w_1 + 2w_2. \) Using the theorem, find the coordinates of \( 5t - 1 \) in basis \( S. \)