Lec 14: Determinants of matrices

Determinants are defined for square matrices only. It is just a number associated with a matrix $A$. This number is denoted by $\det(A)$. By definition, if $A = [a]$ is a $1 \times 1$ matrix, $\det(A)$ is the entry of $A$, i.e. $\det([a]) = a$. To define $\det(A)$ when $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is of order 2, we consider one example which will serve as motivation.

Let’s find the area of a parallelogram $OABC$ generated by vectors $\vec{OA} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\vec{OC} = \begin{bmatrix} c \\ d \end{bmatrix}$ as shown on Figure 1. [Coordinates of points $A$ and $C$ are respectively $(a, b)$ and $(c, d)$.

![Figure 1: Parallelogram generated by vectors $\vec{OA}$ and $\vec{OC}$](image)

The area of $OABC$ is twice the area of the triangle $OAC$, and

$$\text{Area}(OAC) = \text{Area}(OPC) + \text{Area}(PQAC) - \text{Area}(OQA)$$

(1)

We shall use the formulas for areas of triangles and trapezoids. We have

$$\text{Area}(OPC) = \frac{|OP||PC|}{2} = \frac{cd}{2}, \quad \text{Area}(OQA) = \frac{ab}{2},$$

$$\text{Area}(PQAC) = |PQ| \frac{|CP| + |AQ|}{2} = (a - c) \frac{d + b}{2}.$$ 

Plugging these to formula (1), we obtain after simple manipulations: $\text{Area}(OAC) = \frac{ad - bc}{2}$, hence

$$\text{Area}(OABC) = ad - bc.$$ 

(0.1)

Note that the number $ad - bc$ may be negative. It is positive for our picture, but if $\vec{OA}$ were above $\vec{OC}$, then it would be negative. So to be correct we should have
written \(|ad - bc|\) because areas are positive. Without taking absolute value, we have not area but oriented area which may be negative.

For a matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]

we define

\[
det(A) = a_{11}a_{22} - a_{12}a_{21}.
\]

By the example above, \(det(A)\) is the (oriented) area of a parallelogram generated by columns of \(A\).

A 3 × 3 matrix

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

has the determinant

\[
det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.
\]

(3)

Similarly, one can show that this is the area of a parallelepiped generated by columns of \(A\). [Don’t try to do this!] Remember formulas (2) and (3). A simple way to remember formula (3) will be shown in class.

For example,

\[
det([3]) = 3, \quad det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1 \cdot 4 - 3 \cdot 2 = -2,
\]

\[
det\left(\begin{bmatrix} 0 & -1 & 2 \\ 1 & 3 & 4 \\ -2 & 5 & 0 \end{bmatrix}\right) = 0 \cdot 3 \cdot 0 + (-1) \cdot 4 \cdot (-2) + 1 \cdot 5 \cdot 2 - 2 \cdot 3 \cdot (-2) - 0 \cdot 4 \cdot 5 - (-1) \cdot 1 \cdot 0 = 30.
\]

Hence the area of a parallelogram generated by vectors \(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\) and \(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\) is \(|-2| = 2\) square units. The area of a parallelepiped generated by columns of the last matrix is \(|30| = 30\) cubic units.

Before giving a general definition of the determinant, let’s talk about permutations. A permutation on the set \(S = \{1, 2, \ldots, n\}\) is a rearrangement \(j_1j_2\ldots j_n\) of its elements. Here \(j_1\) is a number where 1 goes, \(j_2\) is the image of 2, etc. For example, \(2431\) is a permutation on \(S = \{1, 2, 3, 4\}\) which sends 1, 2, 3, 4 to 2, 4, 3, 1 respectively \((j_1 = 2, j_2 = 4, j_3 = 3, j_4 = 1)\). In particular, it does not move element 3. Permutation \(12\ldots n\) (i.e. \(j_i = i\)) preserves all elements on their places. It is called trivial. Of course, all numbers in the permutation \(j_1j_2\ldots j_n\) are different since they are different in \(S\). How many permutations are there for the set \(S\)? Try to create a permutation. We can move 1 to any number, so there are \(n\) options for \(j_1\). After that, \(j_2\) can be any different from \(j_1\), hence there are \(n - 1\) options for \(j_2\). Number \(j_3\) can be any but \(j_1\) and \(j_2\) because those places are already occupied. We conclude that there are \(n - 2\) possibilities to choose \(j_3\). At this stage, we have \(n(n - 1)(n - 2)\) options for the starting 3
numbers \(j_1j_2j_3\) of our permutation. Finally, there are \(n(n-1)(n-2)(n-3)\cdots 2 \cdot 1 = n!\) variants for the whole permutation \(j_1j_2\ldots j_n\). We conclude that there are exactly \(n!\) permutations on the set of \(n\) elements. The set of all permutations on \(S\) is denoted by \(S_n\).

If a larger number \(j_r\) precedes a smaller one \(j_s\), then we say that elements \((j_r, j_s)\) are an inversion. A permutation \(j_1j_2\ldots j_n\) is called even, if it has even number of inversions, and odd, if the number of its inversions is odd. One can show that the amounts of even and odd inversions are the same, i.e. equal to \(\frac{n!}{2}\) (if \(n > 1\)). The permutation 2431 has 4 inversions: \((2, 1), (4, 3), (4, 1)\) and \((3, 1)\), hence it is even. The permutation 231 is odd, because it has 3 inversions: \((2, 3), (2, 1)\) and \((3, 1)\) (in this case all pairs of elements are inversions). The trivial permutation is even since it contains 0 inversions. The permutation 21345\ldots n (switching 1 and 2) is odd for it only contains one inversion \((2, 1)\). In general, if a permutation transposes only two numbers and leaves the rest on their places, it is odd. [Try to explain it.]

Now let \(A = [a_{ij}]\) be an \(n \times n\) matrix. The determinant of \(A\) is

\[
\det(A) = \sum (\pm) a_{1j_1}a_{2j_2}\cdots a_{nj_n}
\]

where the summation is over all permutations \(j_1j_2\ldots j_n\). We take the sign \(+\) before the term \(a_{1j_1}a_{2j_2}\cdots a_{nj_n}\) if the permutation \(j_1j_2\ldots j_n\) is even, and \(-\) if the permutation is odd. In case \(n = 1\) we have \(\det(A) = a_{11}\). For \(n = 2\) we obtain formula (2) as \(S_2\) contains only two permutations: 12 and 21, the latter being odd. A little bit harder to verify (do it!) that for \(n = 3\) formula (4) is the same as (3). You may regard \(\det\) as a volume of a parallelepiped in space of many dimensions.

Formula (4) contains \(n!\) terms in the sum since it is over all permutations. So for larger \(n\), i.e. \(n = 4, 5, 6, \ldots\) we have 24, 120, 720, \ldots terms to sum. This is extremely boring. On the next lecture we will discover some properties of determinants which will allow to compute them faster.