Math 433 Prelim 1 Solutions Fall 2006

Problem 1 a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix}$ is clearly what we're looking for.

b) Let $V_{/\mathbb{F}}$ be three dimensional. Define $T : V \to V$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y \\ z \\ 0 \end{bmatrix}$. Then $T^2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$ and $T^3\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

c) We simultaneously solve $x + y + z = 0$ and $x + z = 0$. A basis for the solution space is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ so $T : \mathbb{F}^3 \to \mathbb{F}$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x - z$ is what we want.

Problem 2 a) Every basis of $\mathbb{C}$ over $\mathbb{C}$ is one dimensional. We'll take $\mathcal{B} = \{1\}$. Then $[T]_{\mathcal{B}} = a_0 + ib_0$.

b) Every basis of $\mathbb{C}$ over $\mathbb{R}$ is two dimensional. We'll take $\mathcal{B} = \{1, i\}$. Then $[T]_{\mathcal{B}} = \begin{bmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{bmatrix}$.

c) Any linear transformation of the vector space $\mathbb{C}$ over the field $\mathbb{C}$ *must* be multiplication by some complex number $a_0 + ib_0$. When we view $\mathbb{C}$ as a vector space over $\mathbb{R}$, we need only know where to send the basis vectors and make sure this isn't 'multiplication by some complex number'. For instance the transformation that sends 1 to 1 and i to $-i$, that is complex conjugation, is a real linear transformation of $\mathbb{C}$, but not a complex linear transformation of $\mathbb{C}$ as its matrix is with respect to the basis $\{1, i\}$ is $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ which is not of the form in b).

Problem 3 a) Consider the zero transformation $T : \mathbb{F}^4 \to \mathbb{F}^5$. It has highly nontrivial kernel.

b) Let V be the vector space of all polynomials with real coefficients. As there is no bound on the degrees, V is infinite dimensional. Clearly the linear transformation $D : V \to V$ given by $D(f) = df/dx$ is surjective. Its kernel is the set of constant functions.

c) Recall that is an $n \times n$ matrix X is invertible then when viewed as a linear transformation from $\mathbb{F}^n \to \mathbb{F}^n$ that X is one-to-one and onto. If $SAS^{-1} = B$, when viewed as a linear transformations from $\mathbb{F}^n \to \mathbb{F}^n$ both SA and S have kernels of the same dimension. the same applies for SA and SAS^{-1}. (We are using that S and S^{-1} are invertible). Thus $SAS^{-1} = B$ and A have the same dimension kernel. By the rank-nullity theorem their ranks are the same.

Problem 4 a) Let $V = \mathbb{R}$, the one dimensional vector space over $\mathbb{R}$. Define $B : V \times V \to \mathbb{R}$ by $B(\alpha, \beta) = \alpha\beta$. A less trivial example is with $V = \mathbb{R}^2$ and we define $B : V \times V \to \mathbb{R}$ by

$B\left(\begin{bmatrix}\alpha\\ \gamma\end{bmatrix}, \begin{bmatrix}\beta\\ \delta\end{bmatrix}\right) = \alpha\delta - \beta\gamma.$
b) Let $V = \mathbb{R}^2$ and define $B : V \times V \to \mathbb{R}$ by $B(\alpha\vec{e}_1 + \gamma\vec{e}_2, \beta\vec{e}_1 + \delta\vec{e}_2) = \gamma\delta$. Why does this work?

Problem 5 Suppose first that $\text{Ker}(T) = \text{Image}(T) = n$. By the rank-nullity theorem we have $\dim V = n + n = 2n$. Note that the rank of a matrix for T is the dimension of the image of $T = n = \dim(V)/2$. Let $\vec{v} \in V$. Then $T(\vec{v}) \in$Image$(T) = \text{Ker}(T)$, so $T(T(\vec{v})) = \vec{0}$. Thus $T^2 = 0$.

Now suppose $\dim(V)$ is even, $T^2 = 0$ and the rank of a matrix for T is $\dim(V)/2$. We can write $\dim(V) = 2n$ and the dimension of $\text{Image}(T) = n$. Let $\vec{x} \in$Image(T). Then $\vec{x} = T(\vec{v})$ for some $\vec{v} \in V$. As $T^2 = 0$, we see $T(T(\vec{v})) = T(\vec{x}) = \vec{0}$ so $\vec{x} \in \text{Ker}(T)$. Thus $\text{Image}(T) \subseteq \text{Ker}(T)$, so $\dim(\text{Ker}(T)) \geq n$. By the rank-nullity theorem we have $n + \dim(\text{Ker}(T)) = 2n$ so we conclude $\dim(\text{Ker}(T)) = n$ and $\text{Image}(T) = \text{Ker}(T)$.

Problem 6 (b) Set $\vec{y} = \sum_{i=1}^{p} (\vec{x}, \vec{v}_i)\vec{v}_i$. We claim $(\vec{x} - \vec{y}, \vec{y}) = 0$. Indeed

$$(\vec{x}-\vec{y},\vec{y}) = (\vec{x},\vec{y}) - (\vec{y},\vec{y}) = \sum_{i=1}^{p} |(\vec{x},\vec{v}_i)|^2 - (\vec{y},\vec{y}) = \sum_{i=1}^{p} |(\vec{x},\vec{v}_i)|^2 - \sum_{j=1}^{p}\sum_{k=1}^{p} (\vec{x},\vec{v}_j)(\vec{x},\vec{v}_k)(\vec{v}_j,\vec{v}_k)$$

$$= \sum_{i=1}^{p} |(\vec{x},\vec{v}_i)|^2 - \sum_{j=1}^{p} |(\vec{x},\vec{v}_j)|^2 = 0.$$

So $\vec{x} = (\vec{x} - \vec{y}) + \vec{y}$ where $\vec{x} - \vec{y}$ and $\vec{y}$ are orthogonal. Thus

$$||\vec{x}||^2 = ((\vec{x}-\vec{y})+\vec{y}, (\vec{x}-\vec{y})+\vec{y}) = ||\vec{x}-\vec{y}||^2 + ||\vec{y}||^2 = ||\vec{x}-\vec{y}||^2 + \sum_{i=1}^{p} |(\vec{x},\vec{v}_i)|^2 \geq \sum_{i=1}^{p} |(\vec{x},\vec{v}_i)|^2.$$

Problem 7 We'll just do (b). Suppose $\{\vec{v}_1, ..., \vec{v}_k\}$ is a linearly dependent set. Then there is at least one nontrivial dependence relation. Measure the *length* of a nontrivial dependence relation by the number of nonzero coefficients in the equation. Then there is a nontrivial dependence relation of minimal length, say $\sum_{j=1}^{r} \beta_j \vec{v}_j = \vec{0}$. Without loss of generality we can assume $\beta_1, \beta_2 \neq 0$. (Why?) Multiplying by α_1 we get

$$\alpha_1\beta_1\vec{v}_1 + \alpha_1\beta_2\vec{v}_2 + ... + \alpha_1\beta_k\vec{v}_k = \vec{0}.$$

Applying T to our minimal dependence relation gives

$$\alpha_1\beta_1\vec{v}_1 + \alpha_2\beta_2\vec{v}_2 + ... + \alpha_k\beta_k\vec{v}_k = \vec{0}.$$

Subtracting the first equation from the second gives

$$(\alpha_2 - \alpha_1)\beta_2\vec{v}_2 + ... + (\alpha_k - \alpha_1)\beta_k\vec{v}_k = \vec{0}.$$

This is a dependence relation. As $\alpha_1 \neq \alpha_2$ and $\beta_2 \neq 0$ we see it is nontrivial, and of less than minimal length. This is a contradiction, so our assumption that a dependence relation exists was wrong. The set $\{\vec{v}_1, ..., \vec{v}_k\}$ is a linearly independent.