Math 433, Fall 2006

Homework 5, Selected Solutions

1) We recall that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases of a finite dimensional vector space U and $T : U \to U$ is a linear transformation, then $[T]_{\mathcal{B}_2} = S^{-1}[T]_{\mathcal{B}_1}S$ where S is the matrix whose columns express $\mathcal{B}_2$ in terms of $\mathcal{B}_1$. Then we have, using that $\det(CD) = \det(DC)$ that $\det([T]_{\mathcal{B}_2}) = \det(S^{-1}[T]_{\mathcal{B}_1}S) = \det(SS^{-1}[T]_{\mathcal{B}_1}) = \det([T]_{\mathcal{B}_1})$. This proves that the determinant of T is well-defined. On to the problem.

Let $\{\vec{w}_1, ...\vec{w}_r\}$ be a basis of W. Augment this to $\mathcal{B} = \{\vec{w}_1, ...\vec{w}_r, ...\vec{w}_n\}$ a basis of V. Since $T(W) \subseteq W$, we see that for $1 \leq i \leq r$ that $T(\vec{w}_i)$ is a linear combination of $\{\vec{w}_1, ...\vec{w}_r\}$. This tells us that $T_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where A is $r \times r$, B is $r \times n - r$, C is $n - r \times n - r$ and the 0 indicates the $n - r \times r$ matrix with all entries 0. Exercise 16.3 shows $\det(T) = \det(A)\det(C)$. We'll show $\det(A) = \det(T|_W)$ and $\det(C) = \det(\tilde{T})$. This completes the proof.

But A is clearly the matrix for $[T]|_W$ so $\det(A) = \det(T|_W)$. It remains to show $\det(C) = \det(\tilde{T})$. It suffices to show that C is the matrix for $\tilde{T}$ with respect to a suitable basis. I claim $\{[\vec{w}_{r+1}], ..., [\vec{w}_n]\}$ is a basis of V/W. Suppose $\sum_{i=r+1}^{n} \alpha_i[\vec{w}_i] = \vec{0}$ in V/W This is equivalent to $\sum_{i=r+1}^{n} \alpha_i\vec{w}_i \in W$, that is to $\sum_{i=r+1}^{n} \alpha_i\vec{w}_i = \sum_{i=1}^{r} \beta_1\vec{w}_i$. As $\{\vec{w}_1, ...\vec{w}_r, ...\vec{w}_n\}$ is a basis of V, we see all the α's and β's are 0. Thus $\{[\vec{w}_{r+1}], ..., [\vec{w}_n]\}$ are independent in V/W. It is almost the definition of V/W that the $\{[\vec{w}_1], ..., [\vec{w}_n]\}$ span V/W. But the first r of these are $\vec{0}$ in V/W so $\{[\vec{w}_{r+1}], ..., [\vec{w}_n]\}$ spans V/W and is a basis of V/W. We call this basis $\mathcal{C}$. It suffices to show $\tilde{T}_{\mathcal{C}} = C$. But $\tilde{T}[\vec{w}_i] = [T(\vec{w}_i)]$ and for $r + 1 \leq i \leq n$ we have $T(\vec{w}_i) \sum_{j=1}^{r} B_{i-r,j}\vec{w}_j + \sum_{j=r+1}^{n} C_{i-r,j}\vec{w}_j$ where $B_{i-r,j}$ and $C_{i-r,j}$ are the corresponding entries of B and C. Then $\tilde{T}[\vec{w}_i] = [T(\vec{w}_i)] = [\sum_{j=r+1}^{n} C_{i-r,j}\vec{w}_j]$ so $\tilde{T}_{\mathcal{C}} = C$ and we are done.

2) a) Let G and H be k-multilinear on V. It suffices to show that for any $\alpha \in \mathbb{F}$ that $G + \alpha H$ is k-multilinear on V. Fixing all entries but the ith, we see

$$(G + \alpha H)(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u} + \beta\vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n)$$

$$\begin{aligned} &= G(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u} + \beta\vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n) + \alpha H(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u} + \beta\vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n)i \\ &= G(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u}, \vec{v}_{i+1}, ..., \vec{v}_n) + \beta G(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n) \\ &\quad + \alpha H(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u}, \vec{v}_{i+1}, ..., \vec{v}_n) + \alpha\beta H(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n) \\ &= (G + \alpha H)(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{u}, \vec{v}_{i+1}, ..., \vec{v}_n) + \beta(G + \alpha H)(\vec{v}_1, \vec{v}_2, ..., \vec{v}_{i-1}, \vec{w}, \vec{v}_{i+1}, ..., \vec{v}_n). \end{aligned}$$

The second equality follows by multilinearity of G and H. The last line establishes the k-multilieearity of $(G + \alpha H)$.

b) Let $\{\vec{v}_1, ..., \vec{v}_n\}$ be a basis of V. For each pair of integers (r, s) with $1 \leq r, s \leq n$ define $\Psi_{rs} : V \times V \to \mathbb{F}$ by $\Psi_{rs}(\sum_{i=1}^{n} \alpha_i\vec{v}_i, \sum_{j=1}^{n} \beta_j\vec{v}_j) = \alpha_r\beta_s$. It is easy to see each Ψ_{rs} is 2-multilinear. We prove they are independent. Suppose $\sum_{r,s} \gamma_{rs}\Psi_{rs} = 0$. Applying the right hand side to the pair $(\vec{v}_r, \vec{v}_s)$ yields $\gamma_{rs} = 0$. As r and s were arbitrary this shows all coefficients are zero and establishes independence.

We now show the Ψ_{rs} span the 2-multilinear functions. Let Ψ be 2-multilinear and set $\gamma_{rs} = \Psi(\vec{v}_r, \vec{v}_s)$. We claim $\Psi = \sum \gamma_{rs}\Psi_{rs}$. Indeed

$$\Psi(\sum_{i=1}^{n} \alpha_i \vec{v}_i, \sum_{j=1}^{n} \beta_j \vec{v}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\beta_j\Psi(\vec{v}_i, \vec{v}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\beta_j\gamma_{ij}$$

while

$$\sum_{r,s} \gamma_{rs}\Psi_{rs}(\sum_{i=1}^{n} \alpha_i \vec{v}_i, \sum_{j=1}^{n} \beta_j \vec{v}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{rs} \gamma_{rs}\alpha_i\beta_j\Psi_{rs}(\vec{v}_i, \vec{v}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\beta_j\gamma_{ij}.$$

So the Ψ_{rs} form a basis for the vector space of 2-multilinear functions on V. This space is n^2 dimensional.