Math 433 **first prelim** **Wednesday 28 September 2005** **12:20-1:10 pm**

You may keep this paper. Do not return it with your exam booklet.

Show your work. To earn full credit you must give complete and well-reasoned solutions, written up in coherent English. You may use anything that has been given in class, in the homework, or in the book, as long as you show clearly what you are using. You may use a one-sided letter-size cheat sheet. No books, notes, or calculators allowed.

Solutions will be available from the course website after the exam.

1 (20 points). Let $\mathbf{F}$ be the field $\mathbf{F}_3$ of three elements and let $V = \mathbf{F}^3$. Define a linear operator f from V to itself by $f((a,b,c)^T) = (2a+c, a+2b, a+b+c)^T$.

(a) Find the general solution, in vector form, of the equation $f(x) = y$, where $y = (1,1,0)^T$. Don't forget the field is $\mathbf{F}_3$, not $\mathbf{R}$!
(b) Find a basis for the kernel of f.
(c) Find a basis for the image of f.
(d) The triple $\big((1,0,0)^T, (2,1,0)^T, (0,2,1)^T\big)$ is an ordered basis $\mathcal{B}$ of V.
(e) Find the matrix of f with respect to the basis $\mathcal{B}$.

2 (20 points). Let $W_1, W_2, \dots, W_n$ be linear subspaces of a vector space V over a field $\mathbf{F}$. Define $W = \{\, w_1 + w_2 + \cdots + w_n \mid w_1 \in W_1,\ w_2 \in W_2, \dots,\ w_n \in W_n \,\}$.

(a) The map $f\colon \bigoplus_{i=1}^{n} W_i \to V$ defined by $f((w_1, w_2, \dots, w_n)^T) = w_1 + w_2 + \cdots + w_n$ is linear.
(b) W is a linear subspace of V.
(c) The following statements are equivalent:
  (i) f is injective.
  (ii) For all $w \in W$ there exist unique elements $w_1 \in W_1$, $w_2 \in W_2, \dots, w_n \in W_n$ such that $w = w_1 + w_2 + \cdots + w_n$.
  (iii) If $w_1 \in W_1$, $w_2 \in W_2, \dots, w_n \in W_n$ are vectors satisfying $w_1 + w_2 + \cdots + w_n = 0$, then $w_1 = w_2 = \cdots = w_n = 0$.

To avoid wasting time, note that it suffices to show that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i).

3 (20 points). Let V be a vector space over a field $\mathbf{F}$ and let V^* be its dual. For subsets $A \subseteq V$ and $B \subseteq V^*$ we define the *annihilator* A°, resp. B°, by

$$A^\circ = \{\, l \in V^* \mid l(a) = 0 \text{ for all } a \in A \,\},$$
$$B^\circ = \{\, v \in V \mid b(v) = 0 \text{ for all } b \in B \,\}.$$

(a) A° and B° are linear subspaces of V^*, resp. V, for all subsets $A \subseteq V$ and $B \subseteq V^*$.
(b) If W is a linear subspace of V, then $(W^\circ)^\circ = W$. (First show $W \subseteq (W^\circ)^\circ$. For the reverse inclusion, suppose there exists a $v \in (W^\circ)^\circ$ which is not in W, and argue by contradiction.)
(c) Suppose $\dim V = n < \infty$. If W is a linear subspace of V, then

$$\dim W + \dim W^\circ = n.$$

(Start by choosing a basis $\{v_1, v_2, \dots, v_n\}$ of V such that $\{v_1, v_2, \dots, v_p\}$ is a basis of W, and then find a basis of W°.)