**Math 433** **first prelim** **Wednesday 28 September 2005** **12:20-1:10 pm**

*Solutions*

**1** (20 points). (a) Form the matrix $A$ of $f$ w.r.t. standard basis and the augmented matrix $(A \mid y)$ and row reduce:

$$(A \mid y) = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

General solution: write $x_3 = t$ (nonpivot variable), then $x_1 + 2t = 2$, $x_2 + 2t = 1$, $x_3 = t$, so

$$x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(b) $\ker f = \{\, t(1,1,1)^{\mathsf{T}} \mid t \in \mathbf{F} \,\}$ (general solution of homogeneous system), which is spanned by $(1,1,1)^{\mathsf{T}}$.

(c) Pivot variables are $x_1$ and $x_2$, so column space of $A$ is spanned by first two columns, so image of $f$ is spanned by first two columns of $A$.

(d) Line the vectors up in a matrix. The resulting matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

is in row echelon form and has rank 3, so its columns are independent and span $\mathbf{F}^3$.

(e) Let $v_1 = (1,0,0)^{\mathsf{T}}$, $v_2 = (2,1,0)^{\mathsf{T}}$ and $v_3 = (0,2,1)^{\mathsf{T}}$. Then $f(v_1) = (2,1,1)^{\mathsf{T}} = v_1 + 2v_2 + v_3$, $f(v_2) = (1,1,0)^{\mathsf{T}} = 2v_1 + v_2$, $f(v_3) = (1,1,0)^{\mathsf{T}} = 2v_1 + v_2$, so matrix is

$$B = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

**2** (20 points). (a) By definition of vector space structure on direct sum:

$$f(\lambda(w_1, w_2, \dots, w_n)^{\mathsf{T}} + \lambda'(w_1', w_2', \dots, w_n')^{\mathsf{T}}) = f(\lambda w_1 + \lambda' w_1', \lambda w_2 + \lambda' w_2', \dots, \lambda w_n + \lambda' w_n')$$
$$= (\lambda w_1 + \lambda' w_1') + (\lambda w_2 + \lambda' w_2') + \cdots + (\lambda w_n + \lambda' w_n') = \lambda(w_1 + w_2 + \cdots + w_n) + \lambda'(w_1' + w_2' + \cdots + w_n')$$
$$= \lambda f\big((w_1, w_2, \dots, w_n)^{\mathsf{T}}\big) + \lambda' f\big((w_1', w_2', \dots, w_n')^{\mathsf{T}}\big),$$

so $f$ is linear.

(b) $W$ is the image of the linear map $f$ and is therefore a linear subspace.

(c) (i) $\Longrightarrow$ (ii): if $w = \sum_i w_i = \sum_i w_i'$ with $w_i$, $w_i' \in W_i$, then

$$0 = \sum_i w_i - \sum_i w_i' = \sum_i (w_i - w_i') = f\big((w_1 - w_1', w_2 - w_2', \dots, w_n - w_n')^{\mathsf{T}}\big),$$

so $w_1 = w_1'$, $w_2 = w_2'$, …, $w_n = w_n'$, because $f$ injective.

(ii) $\Longrightarrow$ (iii): taking $w = 0$ in (ii) we get that there exist unique elements $w_i \in W_i$ satisfying the condition $\sum_i w_i = 0$. Hence $w_1 = w_2 = \cdots = w_n = 0$ are the only such elements.

(iii) $\Longrightarrow$ (i): if $f((w_1, w_2, \ldots, w_n)^T) = 0$, then $w_1 + w_2 + \cdots + w_n = 0$, so $w_1 = w_2 = \cdots = w_n = 0$, i.e. $(w_1, w_2, \ldots, w_n)^T = 0$. Hence $\ker f = \{0\}$, so $f$ injective.

**3** (20 points). (a) If $l_1$, $l_2 \in A^\circ$, then for all $a \in A$ and $\lambda_1$, $\lambda_2 \in \mathbf{F}$, $(\lambda_1 l_1 + \lambda_2 l_2)(a) = \lambda_1 l_1(a) + \lambda_2 l_2(a) = 0$, so $\lambda_1 l_1 + \lambda_2 l_2 \in A^\circ$. Also $0 \in A^\circ$, so $A^\circ$ is a subspace. Likewise, if $v_1$, $v_2 \in B^\circ$, then for all $b \in B$ and $\lambda_1$, $\lambda_2 \in \mathbf{F}$, $b(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 b(v_1) + \lambda_2 b(v_2) = 0$, so $\lambda_1 v_1 + \lambda_2 v_2 \in B^\circ$. Also $0 \in B^\circ$, so $B^\circ$ is a subspace.

(b) If $v \in W$, then $l(v) = 0$ for all $l \in W^\circ$. Hence $W \subseteq (W^\circ)^\circ$. Suppose there exists $v \in (W^\circ)^\circ$ which is not in $W$. Choose a basis $\mathcal{B}$ of $W$; then $\mathcal{B} \cup \{v\}$ is independent because $v$ is not in $W$. Extend $\mathcal{B} \cup \{v\}$ to a basis $\mathcal{A}$ of $V$. (Possible by basis theorem.) Define $l \colon \mathcal{A} \to \mathbf{F}$ by $l(v) = 1$ and $l(u) = 0$ for $u \in \mathcal{A} \setminus \{v\}$. Extend $l$ to a linear map $l \colon V \to \mathbf{F}$, i.e. an element of $V^*$. Then $l \in W^\circ$ because $l(u) = 0$ for $u \in \mathcal{B}$, but $l(v) = 1 \neq 0$, which contradicts $v \in (W^\circ)^\circ$. Conclusion: $(W^\circ)^\circ = W$.

(c) Choose a basis $\{v_1, v_2, \ldots, v_n\}$ of $V$ such that $\{v_1, v_2, \ldots, v_p\}$ is a basis of $W$. (Possible by the basis theorem.) Then $v_i^*(v_j) = \delta_{ij} = 0$ for $i \leq p$ and $j > p$, so if we put $W' = \operatorname{span}\{v_{p+1}^*, v_{p+2}^*, \ldots, v_n^*\}$, then $W' \subseteq W^\circ$. Conversely, if $l \in W^\circ$, then by dual basis theorem $l = \sum_i \lambda_i v_i^*$ where $\lambda_i = l(v_i) = 0$ for $i > p$. Hence $l \in W'$. Conclusion: $W^\circ = \operatorname{span}\{v_{p+1}^*, v_{p+2}^*, \ldots, v_n^*\}$. By dual basis theorem, $v_{p+1}^*$, $v_{p+2}^*, \ldots, v_n^*$ are independent, so $\dim W^\circ = n - \dim W$.