

**Math 433, Homework 9, Due October 29, 2007 at 2:00pm**

From Curtis:

§25 5; §26 3, 4, 8

Also:

1) Find one matrix  $A$  with integer coefficients and primes  $p, q$  and  $t$  in  $\mathbb{Z}$  such that when  $A$  is reduced modulo  $p, q$  and  $t$  the matrix has Jordan Canonical Form of type

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \pmod{p}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \pmod{q} \text{ (with } \alpha \not\equiv \beta \pmod{q}\text{)}, \quad \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \pmod{t}$$

over  $\mathbb{F}_p, \mathbb{F}_q$  and  $\mathbb{F}_t$ , respectively.

2) Let  $M$  be an  $n \times n$  matrix with all coefficients equal to 1. Calculate the characteristic and minimal polynomial of  $M$ . Prove that  $M$  is diagonalizable. Find a diagonal matrix  $D$  and an invertible matrix  $S$  such that  $M = SDS^{-1}$ .

3) Let  $V$  be a finite dimensional vector space over a field  $\mathbb{F}$ . Let  $V^* = L(V, \mathbb{F})$  be its dual and let  $V^{**} = L(V^*, \mathbb{F})$  be the dual of  $V^*$ . Show there is a canonical bijective linear transformation (often called a canonical isomorphism)  $C : V \rightarrow V^{**}$  given as follows: For  $\vec{v} \in V$  and  $x \in V^*$  we have  $C(\vec{v})(x) = x(\vec{v})$ . Your job is to show the map above is (i) a linear transformation, (ii) bijective.

4) Let  $V$  be the vector space of all  $n \times n$  matrices over a field  $\mathbb{F}$ . Let  $A_0$  be an  $n \times n$  matrix and define  $T \in L(V, V)$  by  $T(B) = A_0B - BA_0$ . Let  $f \in V^*$  be the trace function. What is  $T^*(f)$ ?

5) A sequence

$$\dots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \xrightarrow{T_{i+2}} \dots$$

of linear transformations between vector spaces is called a *complex* if for all  $i$  the image of  $T_{i-1}$  is contained in the kernel of  $T_i$ . The  $i$ th *cohomology of the complex* is the vector space  $\text{Ker}(T_i)/\text{Image}(T_{i-1})$ . Let

$$\dots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \xrightarrow{T_{i+2}} \dots$$

and

$$\dots \xrightarrow{S_{i-1}} W_i \xrightarrow{S_i} W_{i+1} \xrightarrow{S_{i+1}} W_{i+2} \xrightarrow{S_{i+2}} \dots$$

be two complexes.

A *map between the complexes* is a collection of linear transformations  $u_i : V_i \rightarrow W_i$  such that  $S_i \circ u_i = u_{i+1} \circ T_i$  for all  $i$ . Show that a map between complexes induces a linear transformation from the  $i$ th cohomology of the first complex to the  $i$ th cohomology of the second.