Math 433, Homework 4, Due September 22, 2006

NOTE: This document contains two pages. Please read page 2 for more problems.

From Curtis:
§13 5; §14, 2,3,4; §15, 1,2,4,8,9

Also:
1) Let V be an n dimensional vector space over $\mathbb{R}$ and let $(\cdot,\cdot) : V \to \mathbb{R}$ be an inner product. Let $\{\vec{u}_1, \ldots, \vec{u}_n\}$ be an orthonormal basis of V with respect to $(\ ,\ )$ and define linear maps for $i = 1, \ldots, n$:

$$u_i^* : V \to \mathbb{R}, \quad u_i^*(\vec{v}) = (\vec{v}, \vec{u}_i).$$

Prove that $\{\vec{u}_1^*, \ldots, \vec{u}_n^*\}$ is a basis of $L(V, \mathbb{R})$, the dual space of V.

2) Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$ be a subspace. For $\vec{v}_1, \vec{v}_2 \in V$ we write $\vec{v}_1 \sim_W \vec{v}_2$ if $\vec{v}_1 - \vec{v}_2 \in W$. Look up the definition of the term *equivalence relation* somewhere.
a) Prove that $\sim_W$ is an equivalence relation. We denote the set of equivalence classes by the symbol V/W.
b) For $\vec{v} \in V$ let $[\vec{v}]$ be its equivalence class. Define an addition rule $\tilde{+}$ on V/W by $[\vec{v}]\tilde{+}[\vec{u}] = [\vec{x} + \vec{y}]$ where $\vec{x} \in [\vec{v}]$, $\vec{y} \in [\vec{u}]$ and the last sum is vector addition in V. Prove this addition on equivalence classes is well-defined, that is it does not depend on the choices of $\vec{x} \in [\vec{v}]$ and $\vec{y} \in [\vec{u}]$.
c) Come up with a definition for scalar multiplication on V/W and show it is well-defined.
d) Now prove V/W is a vector space over $\mathbb{F}$.

3) Let V be a vector space over a field $\mathbb{F}$ and let $W \subseteq V$ be a subspace. Let $T \in L(V, V)$. Suppose $T(W) \subseteq W$. Show that $\tilde{T} : V/W \to V/W$ given by $\tilde{T}([\vec{v}]) = [T(\vec{v})]$ is well-defined and an element of $L(V/W, V/W)$.

4) Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $T \in L(V, V)$. Suppose $T \neq 0$ but $T^n = 0$ for some integer n. Show there exists a basis $\mathcal{B}$ of V such that the matrix A of T with respect to $\mathcal{B}$ satisfies $a_{ij} = 0$ if $i \leq j$.

5) Let V be a finitely generated, inner product space and W a subspace. How would you define an inner product on V/W?

6) Consider the standard inner product $(\cdot,\cdot)_{st}$ on $\mathbb{R}^n$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Define $(\cdot,\cdot)_T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $(\vec{v},\vec{u})_T = (T(\vec{v}),T(\vec{u}))_{st}$.

a) Prove that if T is invertible then $(\cdot,\cdot)_T$ an inner product on $\mathbb{R}^n$.

b) Prove that all inner products on $\mathbb{R}^n$ arise from the standard inner product and an invertible T.