Math 433, Homework 3, Due September 15, 2006

From Curtis:
§10, 5, 10; §11, 1,3,4; §12, 2, 5, 7a), c), e); §13, 1, 2, 9

Also:
1) Let $\mathbb{F}_5$ be the field of 5 elements, sometimes written $\mathbb{Z}/5\mathbb{Z}$. Prove that there exists a vector space V over $\mathbb{F}_5$ with a linear transformation $T : V \to V$ such that T, T^2, T^3 and T^4 are not the identity transformation, but T^5 is the identity transformation.

2) Let V be a finitely generated vector space over a field $\mathbb{F}$. What conditions must you put on V for there to exist a nonzero linear transformation $T : V \to V$ such that $T^2 = 0$, that is $T^2 : V \to V$ is the linear transformation that takes all vectors to $\vec{0}$?

3) Let A be a 2×2 matrix over a field $\mathbb{F}$. Define $\mathcal{C}(A)$, the *centralizer* of A, to be the set of all 2×2 matrices B over $\mathbb{F}$ satisfying $AB = BA$. Show that $\mathcal{C}(A)$ is a vector space over $\mathbb{F}$. Find all possible dimensions of $\mathcal{C}(A)$.

4) Let V be a finitely generated vector space over a field $\mathbb{F}$. Regarding $\mathbb{F}$ as one dimensional vector space over itself, we can form the vector space $L(V, F)$, often denoted by V^*.
(a) Let $\{\vec{v}_1, ..., \vec{v}_n\}$ be a basis of V. For each $i = 1, ..., n$ show there is an element $f_i \in V^*$ satisfying $f_i(v_j) = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$.
(b) Show the set $\{f_1, ..., f_n\}$ forms a basis of V^*.

5) Consider the linear transformation T from $\mathbb{R}^4$ to $\mathbb{R}^4$ that consists of first reflecting in the $xw-$plane, then reflecting in the yw-plane and finally projecting to $xyz-$space. What is the matrix for T with respect to the standard basis?