MATH 332 - ALGEBRA AND NUMBER THEORY
FINAL EXAM - PRACTICE

Note: The final exam will be similar to the first and second prelim in format, except that the final exam will be longer (2 hours). Also, the final exam will cover all the material we have covered, with an emphasis on newer material. In order to review, it would be a good idea to go over the first and second prelim and their respective practice tests.

Problem 1. *Find the quadratic irrational numbers which correspond to the continued fractions:*

$$\langle 1, 2, \overline{3} \rangle, \quad \langle 1, \overline{2, 3} \rangle, \quad \langle \overline{1, 2, 3} \rangle.$$

Solution. As an example, let us find the irrational number α with simple continued fraction given by $\langle 1, \overline{2, 3} \rangle$. In particular, α satisfies:

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \ldots}}}}.$$

First, we calculate the value of $\beta = \langle \overline{2, 3} \rangle$:

$$\beta = 2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \ldots}}}} = 2 + \cfrac{1}{3 + \frac{1}{\beta}} = \langle 2, 3, \beta \rangle.$$

In order to write down the continued fraction on the right hand side as one simple fraction, we may use the table of convergents:

k	-1	0	1	2
a_k		2	3	β
p_k	1	2	7	$7\beta + 2$
q_k	0	1	3	$3\beta + 1$

and so $\beta = \frac{7\beta+2}{3\beta+1}$. Hence, $3\beta^2 + \beta = 7\beta + 2$ and $3\beta^2 - 6\beta - 2 = 0$. Since $\beta > 0$ we must have

$$\beta = \frac{6 + \sqrt{36 + 4 \cdot 3 \cdot 2}}{6} = \frac{3 + \sqrt{15}}{3}.$$

Finally, $\alpha = \langle 1, \beta \rangle$ and so:

k	-1	0	1
a_k		1	β
p_k	1	1	$\beta + 1$
q_k	0	1	β

Hence:

$$\alpha = \frac{\beta + 1}{\beta} = \frac{\frac{3+\sqrt{15}}{3} + 1}{\frac{3+\sqrt{15}}{3}} = \frac{6 + \sqrt{15}}{3 + \sqrt{15}} = \frac{-1 + \sqrt{15}}{2}.$$

□

Problem 2. *Find the continued fraction and the first* 5 *convergents of the numbers:*

$$\sqrt{3}, \quad \frac{1+\sqrt{5}}{2}, \quad \sqrt{23}, \quad \sqrt{101}$$

without using a calculator.

Solution. Use the tables to find:

$$\sqrt{3} = \langle 1, \overline{1,2} \rangle, \quad \frac{1+\sqrt{5}}{2} = \langle \overline{1} \rangle, \quad \sqrt{23} = \langle 4, \overline{1,3,1,8} \rangle, \quad \sqrt{101} = \langle 10, \overline{20} \rangle.$$

Then use the other tables to find some convergents. □

Problem 3. *Let q_k be defined as always (i.e. $q_{-1} = 0$, $q_0 = 1$, $q_{k+1} = a_{k+1}q_k + q_{k-1}$). Prove by induction that $q_k \geq 2^{k/2}$ if $k \geq 2$.*

Solution. This is solved problem 9-7 in the book, p. 288. □

Problem 4. *Find the first few terms and the first few convergents of π and e. Find the first few convergents of π and e (the more the merrier). Here you can use a calculator (but not during the test).*

Solution. For a continued fraction of e, see the following problem. The continued fraction of π starts as follows:

$$\pi = \langle 3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \ldots \rangle.$$

The first few convergents can be calculated in the table:

k	-1	0	1	2	3	4
a_k		3	7	15	1	292
p_k	1	3	22	333	355	103993
q_k	0	1	7	106	113	33102

□

Problem 5. *The infinite simple continued fraction of the number e is*

$$e = \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots \rangle$$

(1) *Justify: e is an irrational number.*
(2) *Justify: e is not a quadratic irrational number.*
(3) *Find the first eight convergents of the continued fraction of e.*
(4) *Find the best rational approximation to e having a denominator less than or equal to 536.*

Solution. A rational number has a finite continued fraction, hence e is irrational. A quadratic irrational has a periodic continued fraction, hence e cannot be a quadratic irrational number. To answer the last questions, find the convergents and then use the theorems on the virtues of the convergents. □

Problem 6. *Let d be a positive integer.*

(1) *Show that the continued fraction of $\sqrt{d^2+1}$ is $\langle d, \overline{2d} \rangle$.*
(2) *Find the continued fractions of $\sqrt{101}$, $\sqrt{290}$, $\sqrt{2210}$.*

Proof. Use the table to find the continued fraction of a quadratic irrational, and notice that $[\sqrt{d^2+1}] = d$. Hence:

k	0	1
r_k	0	d
s_k	1	1
a_k	d	$2d$

By a theorem stated in class, the first time that the term $2a_0 = 2d$ appears in the expansion is at the end of the period. Hence $\sqrt{d^2+1} = \langle d, \overline{2d}\rangle$. Hence $\sqrt{101} = \langle 10, \overline{20}\rangle$, $\sqrt{290} = \langle 17, \overline{34}\rangle$ and $\sqrt{2210} = \langle 47, \overline{94}\rangle$. □

Problem 7. *Describe all the solutions of:*

$$x^2 + 3y^2 = 1, \quad x^2 - 3y^2 = 1, \quad x^2 - 23y^2 = 1.$$

Proof. See book, notes or Álvaro. □

Problem 8. *Find the fundamental solution of the following equations, by using continued fractions:*

$$x^2 - 23y^2 = 1, \quad x^2 - 101y^2 = 1, \quad x^2 - 29y^2 = 1.$$

Proof. See book, notes or Álvaro. □

Problem 9. *Use congruences to show that $x^2 - 2006y^2 = -1$ has no solutions in positive integers. Is the length of the period of $\sqrt{2006}$ even or odd?*

Proof. See book, notes or Álvaro. □