MATH 103 - MATHEMATICAL EXPLORATIONS

PRACTICE FOR PRELIM 1 (MONDAY, MARCH 5TH)

When answering the following problems, you must (i) name the theorems you use, and (ii) explain how this theorem applies in the given situation.

Problem 1. In a standard deck of 52 cards:

- What is the smallest number of cards you must draw to guarantee that you will have at least one pais?
- What is the smallest number of cards you must draw to guarantee that you will have 5 cards of one suit?

Remember, a standard deck of cards has four suits with 13 cards in each suit.

Problem 2. Describe the sequence of all Fibonacci numbers and list the first 10 Fibonacci numbers.

Problem 3. Let p be a prime that is larger than 3. Can $p+1$ ever be prime? What about $p+3$?

Problem 4. Are there infinitely many composite numbers? Prove it. Are there infinitely many prime numbers? Prove it too.

Problem 5. Can a power of 12 be equal to a power of 21? In other words, are there natural numbers n and m such that $12^n = 21^m$? Why? Why not?

Problem 6. Let $M = 2 \times 3 \times 4 \times \cdots \times 10 + 1$. Is M necessarily a prime? What can you say about the prime factors of M? Why?

Problem 7. You started a long mathematics exam at $2:00$pm. You were told that you could work as long as you liked. You worked 487 hours straight. At what time of the day did you finish? (Include AM or PM).

Problem 8.

(1) Is the number 12345678 divisible by 2? By 3? By 6? (Recall that a number is divisible by 3 if the sum of its digits is divisible by 3).
(2) Calculate $2^6 \mod 7$.
(3) Is the number $16^{12345678} - 1$ divisible by 7?

Problem 9. Prove that $\sqrt{13}$ is an irrational number.

Problem 10. Define what is meant by "the set A has the same cardinality as set B".

Problem 11. Show that the rational numbers $\mathbb{Q}$ have the same cardinality as the natural numbers $\mathbb{N}$ (by showing an explicit one-to-one correspondence, as in the lecture).

Problem 12. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+$ be the set of non-negative real numbers. Is the correspondence between these two sets $\mathbb{R} \to \mathbb{R}^+$ given by $x \mapsto x^2$ a one-to-one correspondence? Explain your answer.

Problem 13. Following is a list of some decimal numbers corresponding to the first few natural numbers. Describe Cantor's diagonalization argument, describe its purpose, and write down the first five digits of a decimal number that Cantor's argument produces.

$$\begin{aligned}
1 &\mapsto 0.43682340923\ldots \\
2 &\mapsto 0.44444444444\ldots \\
3 &\mapsto 0.28461924891\ldots \\
4 &\mapsto 0.89124912983\ldots \\
5 &\mapsto 0.12930901231\ldots \\
6 &\mapsto 0.00231231435\ldots
\end{aligned}$$

Problem 14. Let S be the set of all real numbers between 0 and 1 with the property that their decimal expansions only have 0's and 7's. For example, the following numbers are elements of S:

$$0.7777007707070777707000\ldots, \quad 0.000000000007000700077700\ldots$$

- Show two rational numbers in S. Explain why they are rational.
- Show two irrational numbers in S. Explain why they are irrational.
- Show that the cardinality of S is not equal to the cardinality of the set of natural numbers.