MATH 103 - MATHEMATICAL EXPLORATIONS

PRACTICE FOR PRELIM 1 (MONDAY, MARCH 5TH)

When answering the following problems, you must (i) name the theorems you use, and (ii) explain how this theorem applies in the given situation.
HINT: You must be able to state precisely the following results: the Pigeonhole Principle, the Unique Factorization Theorem, Euclid's Theorem (on the infinitude of primes), Fermat's Little Theorem, Cantor's Diagonalization Argument.

Problem 1. In a standard deck of 52 cards:

- What is the smallest number of cards you must draw to guarantee that you will have at least one pair?
- What is the smallest number of cards you must draw to guarantee that you will have 5 cards of one suit?

Remember, a standard deck of cards has four suits with 13 cards in each suit.

Solution. Each suit has 13 different cards (from ace to king). Therefore, by the pigeonhole principle, if we have 14 cards, two of them must be the same number card and form a pair (explanation: we have more cards (14) than types of cards (13), thus, two cards should be of the same type). Also, by the pigeonhole principle, since there are 4 different suits, if we have 5 cards we will have at least two of the same suit; if we have $8 + 1 = 9$ cards, we will have at least three of the same suit; if we have $12 + 1 = 13$ cards, we will have at least four of the same suit; and if we have $16 + 1 = 17$ cards, we will have at least five of the same suit. □

Problem 2. Describe the sequence of all Fibonacci numbers and list the first 10 Fibonacci numbers.

Solution. The Fibonacci numbers are $F_1 = 1$, $F_2 = 1$ and the rest are defined by the recursive relation $F_{n+1} = F_n + F_{n-1}$ (or in words, the next Fibonacci number is the sum of the previous two numbers). The first 10 are:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55.$$

□

Problem 3. Let p be a prime that is larger than 3. Can $p + 1$ ever be prime? What about $p + 3$?

Solution. If p is prime and greater than 3 then $p \neq 2$ and p is odd. Then $p + 1$ will be even (and not $= 2$), and therefore not a prime. Also $p + 3$ will be even, and not a prime. □

Problem 4. Are there infinitely many composite numbers? Prove it. Are there infinitely many prime numbers? Prove it too.

Solution. Yes! There are infinitely many composite numbers. For example, 6^n is composite for all natural numbers n, because $6^n = 2^n \times 3^n$.

Yes! There are infinitely many prime numbers. To prove this, use Euclid's proof. Suppose for a contradiction that there exist only finitely many primes $p_1, p_2, \ldots, p_n$, where p_n is the largest of them. Let $\mathcal{N}$ be defined by $\mathcal{N} = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$. Clearly, $\mathcal{N}$ is larger than p_n so it cannot be prime. Hence it must have a prime divisor in the list $p_1, \ldots, p_n$. However, if some prime number p_i divides $\mathcal{N}$, it must also divide $\mathcal{N} - (p_1 \cdot \ldots \cdot p_n)$ since both numbers are multiples of p_i. Therefore p_i divides 1, which is impossible. Thus our first assumption must be false and there exist infinitely many prime numbers. □

Problem 5. Can a power of 12 be equal to a power of 21? In other words, are there natural numbers n and m such that $12^n = 21^m$? Why? Why not?

Solution. No! The reason is the unique factorization theorem: every natural number $n > 1$ is a prime or a product of primes, and the factorization into primes is **unique**. Then, the equation $12^n = 21^m$ can never be true, because $12 = 3 \cdot 4$ and $21 = 3 \cdot 7$. Since the prime 7 appears in the factorization of 21^m but not in the factorization of 12^n, these two factorizations would be distinct, which is impossible. □

Problem 6. Let $M = 2 \times 3 \times 4 \times \cdots \times 10 + 1$. Is M necessarily a prime? What can you say about the prime factors of M? Why?

Solution. M is not necessarily prime. What we can say is that M is not divisible by 2, nor 3, nor 4, nor 5, nor 6, nor 7, nor $8, 9$, or 10. The reason is that the remainder of division of M by any of those numbers is 1, thus M is not divisible by them. □

Problem 7. You started a long mathematics exam at $2:00$pm. You were told that you could work as long as you liked. You worked 487 hours straight. At what time of the day did you finish? (Include AM or PM).

Solution. You started the exam at 2pm which we will write as the 14th hour. After 487 hours, it will be the 501th hour. In order to find out what time this is, we need to calculate 501 modulo 24:

$$501 \equiv 24 \times 20 + 21 \equiv 21 \mod 24.$$

Hence, the time you finished was the 21st hour, i.e. 9pm. □

Problem 8. (1) Is the number 12345678 divisible by 2? By 3? By 6? (Recall that a number is divisible by 3 if the sum of its digits is divisible by 3).
(2) Calculate $2^6 \mod 7$.
(3) Is the number $16^{12345678} - 1$ divisible by 7?

Solution. (1) Yes! 12345678 is clearly divisible by 2 because it is even; it is divisible by 3 because the sum of the digits is 36, which is a multiple of 3; and it is divisible by 6 because it is divisible by 2 and 3.

(2) 2^6 equals 64, hence $2^6 \equiv 64 \equiv 1 \mod 7$ (which is what Fermat's Little theorem predicts!).

(3) In order to know if a number $N = 16^{12345678} - 1$ is divisible by 7 is enough to check if $N \equiv 0 \mod 7$. Now, in order to calculate $16^{12345678} \mod 7$, we can reduce 16 first modulo 7 (and $16 \equiv 2 \mod 7$). And we know that 12345678 is divisible by 6, so $12345678 = 6 \times 2057613$:

$$16^{1234568} \equiv 2^{12345678} \equiv 2^{(6 \times 2057613)} \equiv (2^6)^{2057613} \equiv 1^{2057613} \equiv 1 \mod 7$$

Finally:

$$N = 16^{12345678} - 1 \equiv 1 - 1 \equiv 0 \mod 7$$

and N is divisible by 7. □

Problem 9. Prove that $\sqrt{13}$ is an irrational number.

Solution. Notice first that 13 is a prime. Suppose that $\sqrt{13}$ was a rational number $\sqrt{13} = \dfrac{n}{m}$ (and assume we have expressed it as a reduced fraction, so n and m are natural numbers with no common factors). Then it is also true that $13 = \dfrac{n^2}{m^2}$ and $13m^2 = n^2$. Thus, n^2 is a multiple of 13 and n must also be a multiple of 13. Let's say $n = 13a$ for some other natural number a. Then:

$$13m^2 = n^2 = (13a)^2 = 13^2a^2$$

and so, $13m^2 = 13^2a^2$. By cancelling 13 on both sides we obtain $m^2 = 13a^2$. Thus m^2 is a multiple of 13 and m must be a multiple of 13.

Now, we assumed that n and m had no common factors, but we have shown that if we assume that $\sqrt{13}$ was rational and equal to n/m then both n and m have a factor of 13. Since this is impossible, we have reached a contradiction, and $\sqrt{13}$ cannot be rational. □

Problem 10. Define what is meant by "the set A has the same cardinality as set B".

Solution. A set A has the same cardinality as a set B if there is a one-to-one correspondence between the elements of A and the elements of B. In other words, there is a correspondence that matches every element of A with exactly one element of B, in such a way that every element of B is also matched with exactly one element of A. □

Problem 11. Show that the rational numbers $\mathbb{Q}$ have the same cardinality as the natural numbers $\mathbb{N}$ (by showing an explicit one-to-one correspondence, as in the lecture).

Solution. See your notes, or the book. □

Problem 12. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+$ be the set of non-negative real numbers. Is the correspondence between these two sets $\mathbb{R} \to \mathbb{R}^+$ given by $x \mapsto x^2$ a one-to-one correspondence? Explain your answer.

Solution. NO! This is not a one-to-one correspondence because the real number 1 is sent to $1^2 = 1$ but also the number -1 is sent to $(-1)^2 = 1$. Similarly, every number x and every number $-x$ are sent to the same positive real number $x^2 = (-x)^2$. □

Problem 13. Following is a list of some decimal numbers corresponding to the first few natural numbers. Describe Cantor's diagonalization argument, describe its purpose, and write down the first five digits of a decimal number that Cantor's argument produces.

$$\begin{aligned}
1 &\mapsto 0.43682340923\ldots\\
2 &\mapsto 0.44444444444\ldots\\
3 &\mapsto 0.28461924891\ldots\\
4 &\mapsto 0.89124912983\ldots\\
5 &\mapsto 0.12930901231\ldots\\
6 &\mapsto 0.00231231435\ldots\\
7 &\mapsto 0.11111111111\ldots\\
8 &\mapsto 0.22222222222\ldots
\end{aligned}$$

Solution. Cantor's argument shows that the real numbers $\mathbb{R}$ and the natural numbers $\mathbb{N}$ are not in a one-to-one correspondence. In order to prove that no such correspondence exists, we suppose its existence and write the first few real numbers which are matched with the first natural numbers. Then we create a real number which cannot be possibly listed in this correspondence, by creating a number which differs from the first real number in the list in its first digit, from the second real number in the list in the second digit, etc. In the case above, Cantor's argument would create a number:

$$0.11111121\ldots$$

by choosing as the nth digit either a 1 or a 2, so that the nth digit of my new number is different from the nth digit of the nth real number in the list. □

Problem 14. Let S be the set of all real numbers between 0 and 1 with the property that their decimal expansions only have 0's and 7's. For example, the following numbers are elements of S:

$$0.77770077070707777707000\ldots, \quad 0.0000000000007000700077700\ldots$$

- Show two rational numbers in S. Explain why they are rational.
- Show two irrational numbers in S. Explain why they are irrational.
- Show that the cardinality of S is not equal to the cardinality of the set of natural numbers.

Solution. A decimal is rational if its expansion is repeating (or terminating), so here are two rational numbers in S:

$$0.7 \ (= 0.700000\ldots = \frac{7}{10}), \quad 0.077077077077077077077\ldots \ (= \frac{77}{999})$$

A decimal expansion which is not repeating must be irrational. So here are two irrational numbers in S:

$$0.707007000700007000007\ldots, \quad 0.770077700077770000777770000007\ldots$$

Finally, we show that the cardinality of S is different from the cardinality of $\mathbb{N}$ by showing that no one-to-one correspondence can exist between both sets, using Cantor's argument. Suppose that there was such $1-1$ correspondence, which began like this, for example:

$$\begin{aligned}
1 &\mapsto 0.7070707770070\ldots \\
2 &\mapsto 0.0000707070770\ldots \\
3 &\mapsto 0.7700770070070\ldots \\
4 &\mapsto 0.0007000700070\ldots \\
5 &\mapsto 0.7777777777777\ldots \\
6 &\mapsto 0.0070000000000\ldots \\
7 &\mapsto 0.0000000007777\ldots \\
8 &\mapsto 0.7777777770000\ldots
\end{aligned}$$

Then, we will name an element of S which cannot be possibly in this list. For this, we create an element of S which differs from the nth number in the list in the nth digit. Here is the first few digits of such number:

$$0.07700770\ldots$$

Notice that this number is not the first in the list because they differ in the first digit, it is not the second number in the list because they differ in the second digit, etc. In general, this new number has a 0 as nth digit if the nth digit of the nth number in the list is a 7, and it has a 7 is the nth digit of the nth number in the list is a 0. Note too that this new number is composed only by 7's and 0's, so it is in S, but by construction, is not in the list. Then such list, or one-to-one correspondence cannot exist, and S and $\mathbb{N}$ have distinct cardinalities. $\square$