MATH 103 - MATHEMATICAL EXPLORATIONS

SOLUTIONS FOR FIRST EXAM, MONDAY, MARCH 5TH

Problem 1. [25 points]

(1) One night of Halloween (2002) I gave out 65 pieces of candy to 50 kids. Show that at least one kid got two pieces of candy.
(2) On another night of Halloween (2004) I gave out 530 pieces of candy to 50 kids. Prove that there was at least one greedy kid who took at least 11 pieces of candy.

Solution. The pigeonhole principle says that if we have more pigeons than pigeonholes, then, no matter how we arrange the pigeons, there will be one hole with more than one pigeon. NOTICE that this is not to say that each hole will have a pigeon! One hole could have all the pigeons (and therefore there is one hole with more than one pigeon) and the rest could be empty!

In any case, we have more pieces of candy than kids, so at least one kid should have more than one piece (not all kids necessarily got candy! One could have all pieces of candy, but the conclusion still holds, one has more than 1 piece).

If we gave out 530 pieces, then there are more than 50 groups of 10 pieces of candy, to be given to 50 kids, then one kid got more than 10 pieces of candy, so the kid got at least 11. Again, one kid could have 530 pieces and the rest none! But still, the conclusion is the same: at least one kid has 11 pieces. □

Problem 2. [25 points]

(1) A day in Jupiter only lasts 9 hours (Earth hours). Thus, the clocks in Jupiter only mark the hours: 1 o'clock, 2 o'clock, ... up to 9 o'clock. Suppose it is now 3 o'clock on Earth and 3 o'clock on Jupiter. What time will be on Earth after 3604 hours? And on Jupiter after 3604 hours?
(2) Is the number $5^{100} - 1$ divisible by 11? Why?

Solution. For part (1), on Earth we need to work modulo 12. If the times is 3, after 3604 hours it will be the 3607 hour and:

$$3607 \equiv 3600 + 7 \equiv 12 \cdot 300 + 7 \equiv 7 \mod 12$$

so the time is 7 o'clock. On Jupiter, we need to work modulo 9:

$$3607 \equiv 3600 + 7 \equiv 9 \cdot 400 + 7 \equiv 7 \mod 9$$

so the time is 7 o'clock as well.

For part (2), we can use Fermat's Little Theorem, which in this case says that $5^{10} \equiv 1 \mod 11$. Since $100 = 10 \times 10$ we obtain:

$$5^{100} \equiv (5^{10})^{10} \equiv (1)^{10} \equiv 1 \mod 11$$

and so $5^{100} - 1 \equiv 1 - 1 \equiv 0 \mod 11$. Since the remainder is 0, we conclude that the number is divisible by 11. □

Problem 3. [25 points]

(1) Define what is meant by "the set A has the same cardinality as the set B".
(2) Show that $\mathbb{N}$, the set of natural numbers, and $\mathbb{Q}^+$, the set of all positive rational numbers, have the same cardinality. You need to describe a correspondence.

(3) Briefly describe the purpose of Cantor's diagonalization argument. **Also**, write down the first six digits of a decimal number that Cantor's argument produces, given the following list:

$$\begin{aligned} 1 &\mapsto 0.11111111111\ldots \\ 2 &\mapsto 0.22222222222\ldots \\ 3 &\mapsto 0.33333333333\ldots \\ 4 &\mapsto 0.44444447232\ldots \\ 5 &\mapsto 0.12345555555\ldots \\ 6 &\mapsto 0.66622666116\ldots \end{aligned}$$

Solution. Two sets have the same cardinality if there is a one to one correspondence between them.

In order to provide a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Q}^+$, we can arrange $\mathbb{Q}^+$ as follows:

$$\begin{array}{cccc} 1/1 & 1/2 & 1/3 & \ldots \\ 2/1 & 2/2 & 2/3 & \ldots \\ 3/1 & 3/2 & 3/3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

and then count these in a "zig-zag" way, as follows:

$$1 \mapsto 1/1,\ 2 \mapsto 1/2,\ 3 \mapsto 2/1,\ 4 \mapsto 3/1, (\text{skip } 2/2),\ 5 \mapsto 1/3, \ldots$$

See the book (p. 154) for a more clear picture. Notice that we need to skip the numbers 2/2, 4/2, 3/3 for they are not reduced and we don't want to double-count any number!

The purpose of Cantor's diagonalization argument is to show that the cardinalities of $\mathbb{N}$ and $\mathbb{R}$ are different, i.e. there is no $1-1$ correspondence between the two sets (or their "infinities" are different). In order to do this, we assume that such a correspondence exists and then create a real number which is not any number in the list produced by such correspondence. In order to choose this "new" real number, we create a number which differs from the nth real number in the list in the nth digit. If the list is the one given as an example, then we simply have to take a number which differs with each of the numbers on the nth digit: $0.211111\ldots$ or $0.012345\ldots$ are valid choices. □

Problem 4. [25 points]

(1) Describe the sequence of all Fibonacci numbers and list the first 8 Fibonacci numbers (the first two numbers are 1 and 1).
(2) What's interesting about the ratio of consecutive Fibonacci numbers?
(3) Find at least one Fibonacci number in each picture below. You also need to find at least three different Fibonacci numbers in total. Explain where you found them (or explain what they count, or write numbers in the pictures).

Solution. The Fibonacci numbers are a sequence of numbers F_n which starts with $F_1 = 1$ and $F_2 = 1$ and the next number is produced by adding the previous two numbers: $F_{n+1} = F_n + F_{n-1}$. The first 8 Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13$ and 21. The ratio of consecutive Fibonacci numbers $2/1$, $3/2$, $5/3$, $8/5, \ldots$ approaches the famous Golden Ratio $(1 + \sqrt{5})/2$.

As we have seen during the lecture, Fibonacci numbers appear often in nature. One way is by counting the number of spirals in pinecones, cauliflowers, daisies, etc (in the pictures, one can find the numbers $5, 8, 13$ in this fashion). They also appear as the number of petals in flowers (13 in the daisy). □