PROF. ALLEN KNUTSON'S MATH 4370 MIDTERM, SPRING 2009

1. Let $I = \langle b^2 - cd, bc - d^2 \rangle$ in $\mathbb{C}[b, c, d]$.
a. Give a term order in which this isn't a Gröbner basis, and extend it to one.
b. Give a term order in which this is a Gröbner basis.
c. Find the Hilbert series.

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a. Lex, with initial terms indicated by $g_1 = \underline{b^2} - cd, g_2 = \underline{bc} - d^2$.

$$S(g_1, g_2) = c(b^2 - cd) - b(bc - d^2) = \underline{bd^2} - c^2 d =: g_3$$

$$S(g_1, g_3) = d^2(b^2 - cd) - b(bd^2 - c^2 d) = bc^2 d - cd^3 \sim 0$$

$$S(g_2, g_3) = d^2(bc - d^2) - c(bd^2 - c^2 d) = \underline{c^3 d} - d^4 := g_4$$

$$S(g_1, g_4) \sim 0 \text{ because the leading terms are relatively prime}$$

$$S(g_2, g_4) = c^2 d(bc - d^2) - b(c^3 d - d^4) = bd^4 - c^2 d^3 \sim 0$$

$$S(g_3, g_4) = c^3(bd^2 - c^2 d) - bd(c^3 d - d^4) = bd^5 - c^5 d \sim c^2 d^4 - c^5 d \sim 0$$

So $\{g_1, g_2, g_3, g_4\}$ are a Gröbner basis.

b. We want initial terms indicated by $\underline{b^2} - cd, bc - \underline{d^2}$. One way to do that is to order things by degree (that helps one be sure one's getting a well-order), and then within a given degree order things *in reverse* by degree in c, and then break ties either in favor of b or d (it doesn't matter here).

c. The Hilbert series for I is the same as for $\text{init}(I)$. Obviously it's much easier if we use the second order! Then it's $\frac{1}{(1-t)^3}(1 - t^2 - t^2 + t^{2+2})$, which you could also write as $\frac{1}{(1-t)^3}(1 - t^2)^2$ if you want to emphasize that (g_1, g_2) is a regular sequence.

(If you really wanted to be goofy you could simplify to $\frac{(1+t)^2}{1-t}$. This isn't a particularly good idea.)

2. Find all the monomial ideals in $\mathbb{C}[x, y]$ whose Hilbert function is $h(0) = 1, h(1) = 2, h(2) = 2$ and it stays 2 thereafter.

If we didn't kill anything the Hilbert series would go $1, 2, 3, 4, \ldots$ not $1, 2, 2, \ldots$. So $h_I(0) = h_0(0)$, $h_I(1) = h_0(1)$, $h_I(2) = h_0(2) - 1$, which means we mustn't kill anything in degree 0 or 1 and we must kill something in degree 2. The choices are x^2, xy, y^2.

In each of those cases, we can draw the picture and see that every diagonal thereafter has two remaining monomials. So we mustn't kill anything else. The three monomial ideals are $\langle x^2\rangle, \langle xy\rangle, \langle y^2\rangle$.

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3. Find all the homogeneous ideals with that Hilbert function from #2.

The same analysis says that we have to kill a homogeneous quadratic: $ax^2 + bxy + cy^2$, where not all a, b, c are 0, and rescaling all three gives the same ideal.[1] If we killed anything else (not already in the ideal), then it would make the initial ideal larger, but we already classified those.

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4. True or false: two Gröbner bases for the same ideal must have the same number of elements. If true, prove it. If false, give a counterexample.

False. $\langle x\rangle = \langle x, -5x\rangle$ and both are Gröbner bases.

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5. Let $X \subseteq \mathbb{C}^2$ be the two points $\{(1, 0), (0, 1)\}$.

a) Which of the polynomials $xy, x^2, x + y$ are in I_X?

b) Let f be the polynomial from part (a) that is in I_X. Find a degree 1 polynomial in I_X, and call it g.

b) Use the Nullstellensatz to prove that $I_X = \langle f, g\rangle$.

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a) xy vanishes on those points.

b) The only line going through the points is $x + y - 1 = 0$.

c) Let $I = \langle xy, x + y - 1\rangle$. The only solutions to those equations are the two points, so $V(I) = X$. The Nullstellensatz says $I_{V(I)} = \sqrt{I}$, so together that says $I_X = \sqrt{I}$. Hence what we want to prove is $I = \sqrt{I}$, then invoke the Nullstellensatz.

[1] That is to say, there's a projective plane's worth of possibilities.

First let's get a reduced Gröbner basis (with respect to a homogeneous lex order, say): $\underline{y^2} - y, \underline{x} + y - 1$. If $p^2 \in I$, then $\mathrm{init}(p)^2 \in \mathrm{init}(I) = \langle y^2, x\rangle$. We'd like to conclude that $\mathrm{init}(p) \in \mathrm{init}(I)$.

The only monomial for which that's not true is $\mathrm{init}(p) = y$. But then $p = cy + d$, which defines a horizontal line, and hence can't vanish on both points of X. Contradiction, hence $\mathrm{init}(p) \in \mathrm{init}(I)$.

That means we can pick an $i \in I$ with $\mathrm{init}(i) = \mathrm{init}(p)$, and replace p by $p - i$, which will again have $(p-i)^2 = p^2 + i(i-2p) \in I$, but have smaller leading term. By induction $p \in I$. So $I = \sqrt{I}$.

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6. For any two ideals I, J, prove $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$ (meaning, prove both inclusions).

$$\begin{aligned} r \in \sqrt{I} \cap \sqrt{J} &\iff \exists m \text{ such that } x^m \in I \text{ and } \exists n \text{ such that } x^n \in J \\ &\iff \exists N \text{ such that } x^N \in I, x^N \in J \\ &\iff \exists N \text{ such that } x^N \in I \cap J \\ &\iff x \in \sqrt{I \cap J} \end{aligned}$$

The only interesting step is the second $\iff$. To go $\Rightarrow$, let $N = \max(m, n)$. To go $\Leftarrow$, let $m, n = N$. It's okay that this can increase the numbers used; we only need to show some power works, not find the smallest one.