

PROF. ALLEN KNUTSON'S MATH 4370 FINAL, SPRING 2009

1. Let $I = \langle x^2, xy + 3y^2 \rangle$.

a) Find I 's Hilbert series.

b) Find an ideal J such that $I \subsetneq J \subsetneq \sqrt{I}$.

c) Prove that there don't exist ideals J_1, J_2, J_3 such that $I \subsetneq J_1 \subsetneq J_2 \subsetneq J_3 \subsetneq \sqrt{I}$.

a) This is a Gröbner basis if we make y more expensive than x , so $\text{init } I = \langle x^2, y^2 \rangle$. Then the inclusion-exclusion formula for Hilbert series of monomial ideals is

$$H_{\text{init } I} = \frac{1}{(1-t)^2} (1 - t^2 - t^2 + t^4) = \left(\frac{1-t^2}{1-t} \right)^2 = (1+t)^2 = 1 + 2t + t^2.$$

Of course, it would be easier to note that the only standard monomials are $1, x, y, xy$, and totaling them up we get $1 + 2t + t^2$.

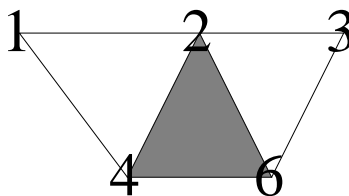
b) First, let's compute \sqrt{I} . Obviously $x \in \sqrt{I} \setminus I$. Adding it in, we get $\langle x^2, xy + 3y^2, x \rangle = \langle 3y^2, x \rangle$. Take that to be J . It's still not radical; y is in $\sqrt{J} \setminus J$. Adding it in, we get $\langle 3y^2, x, y \rangle = \langle x, y \rangle$ which is a squarefree monomial ideal hence radical.

c) $\dim_{\mathbb{C}} R/I = 4$, $\dim_{\mathbb{C}} R/\sqrt{I} = 1$. The (finite) dimensions of R/J_i thus have to be 2 or 3 and there isn't enough room to fit three of them.

2. Let I, J be ideals such that $I \subsetneq J \subsetneq \sqrt{I}$. Prove that J is *not* radical.

We showed long ago that $A \leq B$ implies $\sqrt{A} \leq \sqrt{B}$. So $\sqrt{I} \leq \sqrt{J} \leq \sqrt{\sqrt{I}} = \sqrt{I}$. Hence $\sqrt{I} = \sqrt{J}$. Now since $J \neq \sqrt{I}$, we see $J \neq \sqrt{J}$.

3. Let Δ be the simplicial complex on $\{1, 2, 3, 4, 5, 6\}$ careful! pictured here:



(the only triangle is the $\{2, 4, 6\}$ one; the others are hollow).

a) What are the minimal non-faces of this complex?

b) What is the Hilbert series of its Stanley-Reisner ideal?

Write your answer in the form $p(t)/(1-t)^d$, **in lowest terms**, i.e. $1-t$ doesn't divide the polynomial $p(t)$.

a) $\{5\}, \{1, 3\}, \{1, 6\}, \{3, 4\}, \{1, 2, 4\}, \{2, 3, 6\}$.

b) We have perhaps three ways to do this. One is to look at the 2^6 subsets of the generators. Another is to do inclusion-exclusion. But the easiest is to just add up over all faces (1 empty, 5 vertices, 7 edges, 1 triangle):

$$1 + 5\frac{t}{1-t} + 7\left(\frac{t}{1-t}\right)^2 + 1\left(\frac{t}{1-t}\right)^3 = \frac{1 \cdot (1-t)^3 + 5(1-t)^2 + 7(1-t) + 1}{(1-t)^3}$$

and plainly this numerator is not a multiple of $1-t$.

4. Put the graded lex order on the monomials in $\mathbb{C}[x, y]$, i.e. $1 < y < x < y^2 < xy < x^2 < y^3 < y^2x < yx^2 < x^3 < \dots$

Find all the homogeneous ideals I with $\text{init } I = \langle x^2, xy \rangle$.

Hint: You may assume that any ideal has a unique reduced Gröbner basis, with respect to this term order. (It's true: we just didn't get around to it.)

A reduced Gröbner basis for I will have two (homogeneous) generators, with leading terms x^2 and xy , and no other term being a multiple of x^2 or xy . So that forces the generators to look like

$$x^2 + axy + by^2, \quad xy + cy^2.$$

But are these a Gröbner basis? Compute the S-polynomial:

$$yx^2 - x(xy) \mapsto y(-by^2) - x(-cy^2) = (cy)xy - by^3 \mapsto cy(-cy^2) - by^3 = -(c^2 + b)y^3$$

There we get stuck, saying this isn't a Gröbner basis – unless that's already 0. So the condition is that $c^2 + b = 0$, i.e. the Gröbner bases look like

$$I = \langle x^2 - c^2y^2, xy + cy^2 \rangle.$$

5a) Let X be the ray of positive-real multiples of $(1, 0) \in \mathbb{C}^2$. Compute I_X .

b) Let $I = \langle x^3 - x, xy^2 \rangle$. Draw the vanishing set $V(I) \subseteq \mathbb{C}^2$. Determine the minimal prime ideals P containing I .

a) Plugging in each $(r, 0)$ to $p(x, y)$, we get $p(r, 0) = 0$ for all positive real r . So that polynomial must be the zero polynomial, i.e. every term in $p(x, y)$ had some y in it. Hence $I_X = \langle y \rangle$.

b) The left guy $x^3 - x = 0$ is three vertical lines, with x -coordinates $-1, 0, 1$. The right $xy^2 = 0$ is just the axes. Intersecting, we get the axis $x = 0$ and the two isolated points $(\pm 1, 0)$. Their prime ideals are $\langle x \rangle, \langle x \mp 1, y \rangle$.

6a) Find an ideal $I \leq \mathbb{C}[x]$, such that $\sqrt{I} = \langle x^2 - x \rangle$, and $\dim_{\mathbb{C}} \mathbb{C}[x]/I = 3$.

b) Find *all* such ideals. How many are there?

a,b) Any ideal in $\mathbb{C}[x]$ is generated by one polynomial $p(x)$. We're told that its roots are just the roots of $x^2 - x$, i.e. $0, 1$, and that its degree is 3. So either we put in another factor of x or of $x - 1$, obtaining $I = \langle x^2(x - 1) \rangle$ or $I = \langle x(x - 1)^2 \rangle$.